

ON THE EQUIVARIANT IMPLICIT FUNCTION THEOREM WITH LOW REGULARITY AND APPLICATIONS TO GEOMETRIC VARIATIONAL PROBLEMS

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ABSTRACT. We prove an implicit function theorem for functions on infinite-dimensional Banach manifolds, invariant under the (local) action of a finite dimensional Lie group. Motivated by some geometric variational problems, we consider group actions that are not necessarily differentiable everywhere, but only on some dense subset. Applications are discussed in the context of constant mean curvature hypersurfaces, closed (pseudo-)Riemannian geodesics and harmonic maps.

1. INTRODUCTION

The implicit function theorem is an ubiquitous result from elementary multi-variable calculus courses to current pure and applied research problems. Being the literature on the several formulations of the theorem virtually infinite, we will not attempt to give an account of the extensive variety of statements available. In this paper we will formulate a version of the theorem for functions on Banach manifolds that are invariant under the action of a finite-dimensional Lie group, which is not necessarily compact. Previous formulations of the G -equivariant implicit function theorem, most notably by N. Dancer [8, 9, 10], considered the case of *linear* actions of groups on Banach spaces. Two main improvements are considered here:

- the action is not assumed linear (not even linearizable at the center point);
- the group acts by homeomorphisms of the manifold, and the action is not everywhere differentiable.

Proving an implicit function theorem in such broad context is not just a matter of abstract generality. Namely, as first noticed by R. Palais [21] and others in the 60's, the natural variational framework of several interesting geometric problems involves functionals on Banach manifolds that are invariant under the continuous (but not differentiable) action of a finite-dimensional Lie group of symmetries. Our result is motivated precisely by this type of problem, that includes constant mean curvature (CMC) embeddings, closed geodesics and harmonic maps.

In order to put the problem in the right perspective, let us discuss the CMC variational problem, which is our paradigmatic example. Let M be an m -dimensional compact manifold and $(\overline{M}, \overline{\mathbf{g}})$ be a Riemannian manifold, with $\dim(\overline{M}) = \dim(M) + 1$. Given an embedding $x: M \hookrightarrow \overline{M}$, the *mean curvature* of x is the trace of the

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second fundamental form of x ; and x is a *CMC embedding* when this function is constant. Alternatively, x is a CMC embedding when it is a stationary point of the *area* functional on the space of maps $y: M \hookrightarrow \overline{M}$ of class $\mathcal{C}^{2,\alpha}$, constrained to fixed *volume* variations, i.e., variations by embeddings whose images are boundary of open subsets of \overline{M} having a fixed volume. In this situation, the value of the mean curvature of a constrained critical embedding is (up to a factor) the value of the Lagrange multiplier of this variational problem. The theory of CMC embeddings originated from S. Germain's work on elasticity theory, and plays a pivotal role in a plethora of pure and applied areas. These range from fluid mechanics (where, according to the Young-Laplace equation, mean curvature corresponds to the ratio between the pressure difference and surface tension along the interface between two static fluids) to general relativity (where foliations of asymptotically flat ends of spacetimes by CMC hypersurfaces give a canonical relativistic analog of the Newtonian notion of center of mass), among others.

The CMC constrained variational problem, as posed above, is invariant under the action of two groups: the diffeomorphism group $\text{Diff}(M)$ of the source manifold, acting by right-composition on the space of embeddings; and the isometry group $\text{Iso}(\overline{M}, \overline{\mathbf{g}})$ of the target manifold, acting by left-composition. Two CMC embeddings are *geometrically equivalent* if one can be obtained from the other by a change of parameterization (i.e., by composition on the right with a diffeomorphism of M) and by a rigid motion of the ambient space (i.e., by composition on the left with an isometry of $(\overline{M}, \overline{\mathbf{g}})$). Observe that $\text{Iso}(\overline{M}, \overline{\mathbf{g}})$ is a finite-dimensional Lie group, which may fail to be compact when \overline{M} is not compact. In order to deal with the reparameterization invariance, it is customary to consider a sort of *quotient space* of embeddings of M into \overline{M} modulo diffeomorphisms of M (notice that the right action of $\text{Diff}(M)$ is free). Such quotient space has the structure of an infinite-dimensional topological manifold; natural charts for this set are given around smooth embeddings x by considering the exponential map of the normal bundle x^\perp of x . However, surprisingly enough, the corresponding (local) action of the isometry group $\text{Iso}(\overline{M}, \overline{\mathbf{g}})$ on (a neighborhood of the zero section of) the space of sections of x^\perp is *not* differentiable. Namely, such action involves the operation of taking the inverse of a diffeomorphism: this operation is *not differentiable* (when defined) in the set of $\mathcal{C}^{2,\alpha}$ -diffeomorphisms, but only *continuous*, see [1]. Thus, one has a smooth functional, invariant under the continuous action of a finite-dimensional Lie group, the action of the group being differentiable only on a dense subset. Observe also that the group action in question is essentially nonlinear, in the sense that in general it does not admit fixed points, and no local linearization of the action is feasible.

An analogous situation occurs for the closed geodesic variational problem on (pseudo-)Riemannian manifolds. Here, the group is the circle \mathbb{S}^1 , acting by rotation on the parameter space in the manifold of \mathcal{C}^2 -closed curves. In this case, the action is by diffeomorphisms, but again it is not differentiable at curves with low regularity (in fact, it is differentiable only on the dense subset formed by curves of class \mathcal{C}^3). Other geometric variational problems are invariant under smooth (nonlinear) group actions. This is the case, for instance, of the variational problem for harmonic maps between two Riemannian manifolds. The correspondent energy functional is invariant under the action of the isometry group of the target manifold by left-composition, which is a smooth action.

Our equivariant implicit function theorem is adapted to all of these situations, and involves standard Fredholmness and nondegeneracy assumptions. A rough statement of our main result is as follows.

Theorem. *Let $f: \mathfrak{M} \times \Lambda \rightarrow \mathbb{R}$ be a map of class C^{k+1} , $k \geq 1$, where \mathfrak{M} and Λ are (possibly infinite-dimensional) Banach manifolds, and assume that for all λ , $f(\cdot, \lambda)$ is invariant under the action of a finite-dimensional Lie group G on \mathfrak{M} . Let $(x_0, \lambda_0) \in \mathfrak{M} \times \Lambda$ be such that $\frac{\partial f}{\partial x}(x_0, \lambda_0) = 0$. Assume a suitable Fredholmness hypothesis on $\frac{\partial^2 f}{\partial x^2}(x_0, \lambda_0)$, and the equivariant nondegeneracy of the critical point:*

$$\ker \left(\frac{\partial^2 f}{\partial x^2}(x_0, \lambda_0) \right) = T_{x_0}(G \cdot x_0),$$

where $G \cdot x_0$ denotes the G -orbit of x_0 . Then, there exists a C^k map $x: U \subset \Lambda \rightarrow \mathfrak{M}$, defined in a neighborhood U of λ_0 in Λ , with $x(\lambda_0) = x_0$, such that if $\lambda \in U$ and y is sufficiently close to the orbit $G \cdot x_0$, then $\frac{\partial f}{\partial x}(y, \lambda) = 0$ if and only if y belongs to the G -orbit of $x(\lambda)$.

The precise technical statement of the result (Theorem 3.2) is cast in an abstract Banach vector bundle language, see Section 3. Its formulation involves a set of axioms that describe a rather general setup to which the result applies. Axioms (A) describe the basic variational setup. Axiom (B) deals with the differentiability of the action, and axiom (C) with the G -invariance of the set of critical points. Axioms (D) give the existence of a *gradient-like* map, and axioms (E) guarantee its equivariance. Finally, axioms (F) provide a notion of continuity for the tangent space to the group orbits.

Explicit applications to the above mentioned geometric variational problems are discussed in Section 4; and an extension of the equivariant implicit function theorem to *local* group actions is discussed in Appendix A.

As a sort of motivation for our general result, in Section 2, we discuss a formulation of the G -equivariant implicit function theorem for the specific context of CMC embeddings of closed manifolds. This part of the work originates from an idea of N. Kapouleas [13, 14], also employed by R. Mazzeo, F. Pacard and D. Pollack [19], R. Mazzeo and F. Pacard [20], B. White [24, §3], and, finally, J. Pérez and A. Ros [22, Thm 6.7], who proved the result in the case of embeddings of oriented surfaces in \mathbb{R}^3 . Using a similar idea, we prove a result for arbitrary CMC embeddings $x: M \hookrightarrow \overline{M}$ whose image $x(M)$ is the boundary of a bounded open subset of \overline{M} , see Proposition 2.3. The proof is purely geometric, based on two properties of the flow of Killing vector fields along the boundary of open subsets in Riemannian manifolds, see Lemma 2.1. The curious proof of Proposition 2.3 cannot be extended to the case where $x(M)$ is not the boundary of an open subset of \overline{M} . In particular, this excludes the case of CMC embeddings of manifolds *with boundary*. Our abstract G -equivariant implicit function theorem, applied to this setup, covers a broader situation than Proposition 2.3. Namely, the hypothesis that $x(M)$ is a boundary is replaced with the more general hypothesis that there exists a generalized volume functional in a C^1 -neighborhood of x which is invariant under (small) isometries of the ambient space. Topological and geometrical conditions that guarantee that this hypothesis is satisfied are discussed in Appendix B. For instance, invariant volume functionals exist when the ambient space is diffeomorphic to a sphere or to \mathbb{R}^n ; when the ambient space is not compact but it has compact isometry group (in the

case of embeddings of manifolds with boundary, compactness of the isometry group is not necessary); or when the image of x is contained in an open subset of \overline{M} which has vanishing de Rham cohomology in dimension $m = \dim(M)$.

A natural question is whether the assumption on the existence of an invariant volume functional is redundant for the implicit function theorem in the CMC embedding problem. Namely, observe that such a function is solely needed for the *variational* formulation of the problem, while the *geometric* formulation, i.e., requiring simply that the mean curvature function be constant, is totally independent of the existence of a volume functional. In other words, one could try to study an implicit function theorem for the equation $F(x, H) = \mathcal{H}(x) - H = 0$, where $\mathcal{H}(x)$ is the mean curvature function of the embedding x . The quite surprising answer to this question is that the equivariant implicit function theorem *does not hold* in this general situation without further assumptions.¹ A simple counter-example is obtained by considering closed geodesics (i.e., minimal embeddings of the circle) in the flat 2-torus, which are homotopically non-trivial and nondegenerate in the equivariant sense. This provides a counter-example because there are no homotopically non-trivial circles with constant geodesic curvature in the flat 2-torus, see Example 1. It is also noteworthy that the CMC implicit function theorem holds at a *minimal* embedding $x: M \hookrightarrow \overline{M}$ only under the assumption that x is transversely oriented, see Example 2.

Back to the abstract result, a final technical remark on its proof is in order. The central point is the construction of a sort of *slice*² for the group action at a given smooth critical orbit of the variational problem. More precisely, this is a smooth submanifold S , transversal to the given smooth critical orbit, such that every nearby orbit (not necessarily smooth) intercepts S , and with the property that it is a *natural constraint*, i.e., restriction to S of the variational problem has the same critical points of the non-restricted functional. Given the lack of regularity, transversality at an orbit does not imply non-empty intersection with nearby orbits. The transversality argument is replaced by a topological degree argument, that uses the finite-dimensionality of the group orbits, see Proposition 3.4.

A natural continuation of the ideas developed in this paper include:

- applications of our results regarding rigidity of variational physical models involving CMC hypersurfaces and similar equilibrium interfaces (e.g., anisotropic surface energies and interfaces of liquid crystals with an isotropic substrate);
- an extension of the equivariant implicit function theorem to the case of (continuously) varying groups;
- an analysis of the regularity of the slices for the group actions considered;
- an analysis of possible equivariant bifurcation phenomena.

¹It can be observed that even in the case of linear actions, for the non-variational case, the equivariant implicit function theorem requires some extra hypothesis. For instance, an assumption on the fixed points of the isotropic representation (that the author calls *Property P*) is used in [8, 9, 10].

²The terminology here is not standard. Recall that a “slice” for an action through a point is typically assumed invariant under the action of the isotropy of that point, see [6]. This property is not required here.

2. AN IMPLICIT FUNCTION THEOREM FOR CMC HYPERSURFACES

We will discuss here a version of the implicit function theorem in the context of constant mean curvature (CMC) embeddings in Riemannian manifolds, which serves as motivation for the abstract formulation given in Section 3. This part of the work is inspired by an ingenious idea originally due to N. Kapouleas [13, 14], used also by J. Pérez and A. Ros [22, Thm 6.7], and by B. White [24, §3]. In addition, the same type of argument appears in a paper by R. Kusner, R. Mazzeo and D. Pollack [18, Thm 1.3], and was also recently employed by B. Daniel and P. Mira [11, Prop 5.6] in the study of CMC spheres in the homogeneous manifold Sol_3 . A similar idea is also used in [16] for a deformation result of CMC surfaces in \mathbb{R}^3 with fixed boundary. Both [18] and [22] deal with the case of minimal or CMC immersions of non-compact manifolds.

The basic setup is given by a CMC hypersurface M of a Riemannian manifold \overline{M} . Let us first recall two elementary applications of Stokes' theorem to the computation of integrals involving Killing fields and mean curvature of submanifolds (see [11, Lemma 5.5] for the 2-dimensional orientable case).

Lemma 2.1. *Let $(\overline{M}, \overline{\mathbf{g}})$ be a Riemannian manifold, let $M \subset \overline{M}$ be a compact submanifold (without boundary), with mean curvature vector field \vec{H} , and let $K \in \mathfrak{X}(M)$ be a Killing field in M . Then:*

$$(2.1) \quad \int_M \overline{\mathbf{g}}(K, \vec{H}) \, dM = 0.$$

In addition, if M is the boundary of a (bounded) open subset of \overline{M} , then:

$$(2.2) \quad \int_M \overline{\mathbf{g}}(K, \vec{n}) \, dM = 0,$$

where \vec{n} is a continuous unit normal field along M .

Proof. Denote by $K_M \in \mathfrak{X}(M)$ the vector field on M obtained by orthogonal projection of K . We claim that $\text{div}_M(K_M) = \overline{\mathbf{g}}(K, \vec{H})$. Equality (2.1) will then follow immediately from Stokes' Theorem. In order to compute $\text{div}_M(K_M)$, let $\overline{\nabla}$ denote the Levi-Civita connection of $\overline{\mathbf{g}}$ and let ∇ be the Levi-Civita connection of the induced metric on M . If \mathcal{S} is the second fundamental form of M , then for all pairs $X, Y \in \mathfrak{X}(M)$, one has $\overline{\nabla}_X Y = \nabla_X Y + \mathcal{S}(X, Y)$. Moreover, differentiating in the direction X the equality $\overline{\mathbf{g}}(K_M, Y) = \overline{\mathbf{g}}(K, Y)$, we get:

$$(2.3) \quad \overline{\mathbf{g}}(\nabla_X K_M, Y) + \overline{\mathbf{g}}(K_M, \nabla_X Y) = \overline{\mathbf{g}}(\overline{\nabla}_X K, Y) + \overline{\mathbf{g}}(K, \overline{\nabla}_X Y).$$

Substituting $\overline{\mathbf{g}}(K, \overline{\nabla}_X Y) = \overline{\mathbf{g}}(K, \nabla_X Y) + \overline{\mathbf{g}}(K, \mathcal{S}(X, Y))$ in (2.3),

$$(2.4) \quad \overline{\mathbf{g}}(\nabla_X K_M, Y) = \overline{\mathbf{g}}(\overline{\nabla}_X K, Y) + \overline{\mathbf{g}}(K, \mathcal{S}(X, Y)).$$

Given $x \in M$, an orthonormal frame e_1, \dots, e_m of $T_x M$, and recalling that, since K is Killing, $\overline{\mathbf{g}}(\overline{\nabla}_{e_i} K, e_i) = 0$ for all i , we get:

$$\text{div}_M(K_M) = \sum_i \overline{\mathbf{g}}(\nabla_{e_i} K_M, e_i) = \sum_i \overline{\mathbf{g}}(K, \mathcal{S}(e_i, e_i)) = \overline{\mathbf{g}}(K, \vec{H}),$$

which proves (2.1). Formula (2.2) is an immediate application of Stokes' theorem, observing that $\text{div}_{\overline{M}} K = 0$, as K is Killing. \square

Remark 2.2. It is easy to find counterexamples to (2.2) when M is a hypersurface which is not the boundary of an open subset of \overline{M} . If M is the boundary of an open subset of \overline{M} , i.e., if the set $\overline{M} \setminus M$ has two connected components, then M is *transversely oriented*. This means that the normal bundle TM^\perp is orientable. Conversely, if M is transversely oriented, then the condition that M be the boundary of an open subset of \overline{M} is equivalent to the condition that the homomorphism $H_1(\overline{M}) \rightarrow H_1(\overline{M}, \overline{M} \setminus M)$ induced in singular homology by the inclusion $(\overline{M}, \emptyset) \hookrightarrow (\overline{M}, \overline{M} \setminus M)$ be trivial.

Recall that, given a transversely oriented codimension one CMC embedding $x: M \hookrightarrow \overline{M}$, the *Jacobi operator* J_x of x is the second order linear elliptic differential operator

$$(2.5) \quad J_x(f) = \Delta_x f - (m \operatorname{Ric}_{\overline{M}}(\vec{n}_x) + \|\mathcal{S}_x\|^2)f,$$

defined on the space of \mathcal{C}^2 -functions $f: M \rightarrow \mathbb{R}$. In the above formula, $m = \dim(M)$, Δ_x is the (positive) Laplacian of functions on M relative to the pull-back metric $x^*(\overline{\mathbf{g}})$, $\operatorname{Ric}_{\overline{M}}(\vec{n}_x)$ is the Ricci curvature of \overline{M} evaluated on the unit normal field \vec{n}_x of x and \mathcal{S}_x is the second fundamental form of x . A function f satisfying $J_x(f) = 0$ is called a *Jacobi field* of the embedding x . Moreover, the space of Jacobi fields of x , i.e., the kernel of J_x , is a finite-dimensional space. Given any $\alpha \in]0, 1[$, seen as a linear operator from $\mathcal{C}^{2,\alpha}(M)$ to $\mathcal{C}^{0,\alpha}(M)$, J_x is a Fredholm map³ of index zero, which is symmetric with respect to the L^2 -pairing $\langle \cdot, \cdot \rangle_{L^2}: \mathcal{C}^{2,\alpha}(M) \times \mathcal{C}^{0,\alpha}(M) \rightarrow \mathbb{R}$, given by $\langle f_1, f_2 \rangle_{L^2} = \int_M f_1 \cdot f_2 \, dM$. In particular, $\ker(J_x) = \operatorname{Im}(J_x)^\perp$, relatively to the L^2 -inner product.

If \overline{K} is a Killing field of $(\overline{M}, \overline{\mathbf{g}})$, then $f = \overline{\mathbf{g}}(\overline{K}, \vec{n}_x)$ is a Jacobi field of x . The embedding x is said to be *nondegenerate* if every Jacobi field arises in this way, i.e., if given any Jacobi field f of x , there exists a Killing field \overline{K} of $(\overline{M}, \overline{\mathbf{g}})$ such that $f = \overline{\mathbf{g}}(\overline{K}, \vec{n}_x)$. Nondegeneracy of every CMC embeddings of M into \overline{M} is a *generic* property in the set of Riemannian metrics $\overline{\mathbf{g}}$, see [24, 25] for a precise statement.

Finally, let us introduce the following terminology. Two embeddings $x_i: M \hookrightarrow \overline{M}$, $i = 1, 2$, are said to be *congruent* if there exists a diffeomorphism $\phi: M \rightarrow M$ such that $x_2 = x_1 \circ \phi$, and *isometrically congruent* if there exists a diffeomorphism $\phi: M \rightarrow M$ and an isometry $\psi: \overline{M} \rightarrow \overline{M}$ such that $x_2 = \psi \circ x_1 \circ \phi$. Roughly speaking, congruence classes of embeddings of M into \overline{M} are submanifolds of \overline{M} that are diffeomorphic to M .

Proposition 2.3. *Let $x: M \hookrightarrow \overline{M}$ be a nondegenerate codimension one CMC embedding of a compact manifold M into a Riemannian manifold $(\overline{M}, \overline{\mathbf{g}})$, with mean curvature H_0 . Assume also that $x(M)$ is the boundary of an open subset of \overline{M} . Then, there exists an open interval $]H_0 - \varepsilon, H_0 + \varepsilon[$ and a smooth function $]H_0 - \varepsilon, H_0 + \varepsilon[\ni H \mapsto \varphi_H \in \mathcal{C}^{2,\alpha}(M)$, with $\varphi_{H_0} = 0$, such that:*

- (a) *for all $H \in]H_0 - \varepsilon, H_0 + \varepsilon[$, the map $x_H: M \hookrightarrow \overline{M}$ defined by*

$$x_H(p) = \exp_{x(p)}(\varphi_H(p) \cdot \vec{n}_x(p)), \quad p \in M,$$

is a CMC embedding having mean curvature equal to H ;

³ Second order self-adjoint elliptic operators acting on sections of Euclidean vector bundles over compact manifolds are Fredholm maps of index zero from the space of $\mathcal{C}^{j,\alpha}$ -sections to the space of $\mathcal{C}^{j-2,\alpha}$ -sections, $j \geq 2$, see for instance [24, §1.4] and [25, Theorem 1.1]. This fact will be used throughout the paper.

- (b) any given CMC embedding $y: M \hookrightarrow \overline{M}$ sufficiently close to x (in the $\mathcal{C}^{2,\alpha}$ -topology) is isometrically congruent to some x_H .

Proof. By a standard argument in submanifold theory, congruence classes of embeddings $y: M \hookrightarrow \overline{M}$ near x are parameterized by functions on M . More precisely, to each function $\varphi \in \mathcal{C}^{2,\alpha}(M)$ one associates the map $x_\varphi: M \rightarrow \overline{M}$ defined by $x_\varphi(p) = \exp_{x(p)}(\varphi(p) \cdot \vec{n}_x(p))$, $p \in M$. For φ in a neighborhood of 0, x_φ is an embedding of M into \overline{M} . Conversely, given any embedding $y: M \hookrightarrow \overline{M}$ which is sufficiently close to x , there exists $\varphi \in \mathcal{C}^{2,\alpha}(M)$ near 0 such that y is congruent to x_φ . Given a sufficiently small neighborhood \mathcal{U} of 0 in $\mathcal{C}^{2,\alpha}(M)$, consider the map $\mathcal{H}: \mathcal{U} \rightarrow \mathcal{C}^{0,\alpha}(M)$ that associates to each φ the mean curvature function of the embedding x_φ . This function is smooth, as it is given by a second order quasi-linear differential operator having smooth coefficients. The derivative $d\mathcal{H}(0): \mathcal{C}^{2,\alpha}(M) \rightarrow \mathcal{C}^{0,\alpha}(M)$ coincides with the Jacobi operator J_x .

By the nondegeneracy assumption on x , there exist $d = \dim \ker(J_x) \geq 0$ Killing vector fields $\overline{K}_1, \dots, \overline{K}_d$ of $(\overline{M}, \overline{g})$ such that the functions $f_i = \overline{g}(\overline{K}_i, \vec{n}_x)$, $i = 1, \dots, d$, form a basis of $\ker(J_x)$. Consider now the auxiliary map $\tilde{\mathcal{H}}: \mathcal{U} \times \mathbb{R}^d \rightarrow \mathcal{C}^{0,\alpha}(M)$ defined by:

$$\tilde{\mathcal{H}}(f, a_1, \dots, a_d) = \mathcal{H}(f) + \sum_{i=1}^d a_i f_i.$$

Clearly $\tilde{\mathcal{H}}$ is smooth, and:

$$d\tilde{\mathcal{H}}(0)(f, b_1, \dots, b_d) = J_x(f) + \sum_{i=1}^d b_i f_i.$$

Now, $d\tilde{\mathcal{H}}(0)$ is surjective; namely, the f_i 's span the orthogonal complement of $\text{Im}(J_x)$. Moreover, the kernel of $d\tilde{\mathcal{H}}(0)$ coincides with $\ker(J_x) \oplus \{0\}$, which is finite-dimensional and therefore complemented in $\mathcal{C}^{2,\alpha}(M) \oplus \mathbb{R}^d$. In other words, $\tilde{\mathcal{H}}$ is a smooth submersion at 0.

Using the local form of submersions, we get that for H near H_0 , there exists an open neighborhood \mathcal{V} of 0 in $\mathcal{C}^{2,\alpha}(M) \times \mathbb{R}^d$ such that the set $\tilde{\mathcal{H}}^{-1}(H) \cap \mathcal{V}$ is a smooth embedded submanifold of dimension d . Moreover, using the fact that submersions admit smooth local sections, one has that there exists a smooth function $]H_0 - \varepsilon, H_0 + \varepsilon[\ni H \mapsto \tilde{\varphi}_H \in \mathcal{V}$ such that $\tilde{\mathcal{H}}(\tilde{\varphi}_H) = H$ for all H , and with $\tilde{\varphi}_{H_0} = 0$. Now, we claim that for all $H \in \mathbb{R}$, given $\tilde{\varphi} = (\varphi, a_1, \dots, a_d) \in \tilde{\mathcal{H}}^{-1}(H)$, then $a_1 = \dots = a_d = 0$, and $\mathcal{H}(\varphi) = H$; in other words, $\tilde{\mathcal{H}}^{-1}(H) = \mathcal{H}^{-1}(H) \times \{0\}$. In order to prove the claim, assume

$$\mathcal{H}(\varphi) + \sum_{i=1}^d a_i f_i = H.$$

Multiplying both sides of this equality by $\sum_i a_i f_i$ and integrating on M , keeping in mind that:

$$\int_M \mathcal{H}(\varphi) \sum_{i=1}^d a_i f_i \, dM \stackrel{(2.1)}{=} 0 \quad \text{and} \quad H \cdot \int_M \sum_{i=1}^d a_i f_i \, dM \stackrel{(2.2)}{=} 0,$$

we get:

$$\int_M \left[\sum_{i=1}^d a_i f_i \right]^2 dM = 0.$$

This implies $a_1 = \dots = a_d = 0$ and proves the claim. Hence, we have $\tilde{\varphi}_H = (\varphi_H, 0, \dots, 0)$, with $H \mapsto \varphi_H$ satisfying item (a) of the proposition.

Item (b) also follows easily. Namely, the action by isometries of $(\overline{M}, \overline{\mathbf{g}})$ on each CMC embedding x_{φ_H} produces an orbit which is a d -dimensional submanifold of the Banach space $\mathcal{C}^{2,\alpha}(M)$.⁴ Such orbit is contained in $\mathcal{H}^{-1}(H)$, which is also a d -dimensional submanifold around x_H . Hence, a neighborhood of x_H in the orbit of x_H coincides with a neighborhood of x_H in $\mathcal{H}^{-1}(H)$. This implies that CMC embeddings $\mathcal{C}^{2,\alpha}$ -close to x must be isometrically congruent to some x_H . \square

Remark 2.4. Observe that the assumption that $x(M)$ be the boundary of a bounded open subset of \overline{M} cannot be omitted in Proposition 2.3, as equality (2.2) is used in the proof (see Remark 2.2). In particular, Proposition 2.3 does not cover the case of CMC embeddings of manifolds with boundary (cf. Proposition 4.1).

3. STATEMENT OF THE G -EQUIVARIANT IMPLICIT FUNCTION THEOREM

The usual formulations of the implicit function theorem give a local result, so that its statement can be given using open subsets of Banach spaces as domains and codomains of the functions involved. For the equivariant version of the theorem that will be discussed in this section the situation is somewhat different. Namely, we will consider group actions on Banach manifolds whose orbits are not necessarily contained in the domain of some local chart, or in the domain of a local trivialization of a vector bundle. In fact, we will not even assume boundedness of the orbits. This suggests that, in spite of the local character of the result and its proof, the equivariant formulation of our theorem is better cast in an abstract Banach manifolds/Banach vector bundles setup.

The basic setup is given by a manifold \mathfrak{M} acted upon by a Lie group G , another manifold Λ , and a differentiable function $\mathfrak{f}: \mathfrak{M} \times \Lambda \rightarrow \mathbb{R}$ which is G -invariant in the first variable. More precisely, our framework is described by the following set of axioms.

- (A1) \mathfrak{M} and Λ are differentiable manifolds, modeled on a (possibly infinite-dimensional) Banach space;
- (A2) G is a finite-dimensional Lie group, acting continuously on \mathfrak{M} (on the left) by homeomorphisms, and \mathfrak{g} denotes its Lie algebra;
- (A3) $\mathfrak{f}: \mathfrak{M} \times \Lambda \rightarrow \mathbb{R}$ is a function of class \mathcal{C}^{k+1} , $k \geq 1$, satisfying $\mathfrak{f}(g \cdot x, \lambda) = \mathfrak{f}(x, \lambda)$ for all $g \in G$, $x \in \mathfrak{M}$ and $\lambda \in \Lambda$.

Here, $g \cdot x$ denotes the action of the group element g on x ; and the Lie algebra of G will be denoted by \mathfrak{g} . For all $x \in \mathfrak{M}$, let us denote by

$$(3.1) \quad \beta_x: G \longrightarrow \mathfrak{M}$$

⁴This is not a trivial fact, keeping into account that the left action of the isometry group of $(\overline{M}, \overline{\mathbf{g}})$ on the space $\mathcal{C}^{2,\alpha}(M)$ obtained via exponential map of the normal bundle of x is only continuous, and not differentiable. However, it is proved in [1] that the orbit of any smooth embedding is a smooth submanifold.

the map $\beta_x(g) = g \cdot x$; and for all $g \in G$, the map $\gamma_g: \mathfrak{M} \rightarrow \mathfrak{M}$ will denote the homeomorphism $\gamma_g(x) = g \cdot x$. As to the regularity of the group action, we make the following assumptions:

- (B) there exists a dense subset $\mathfrak{M}' \subset \mathfrak{M}$ such that for all $x \in \mathfrak{M}'$ the map $\beta_x: G \rightarrow \mathfrak{M}$ is differentiable at $1 \in G$.

Let us denote by $\partial_1 \mathfrak{f}: \mathfrak{M} \times \Lambda \rightarrow T\mathfrak{M}^*$ the derivative of \mathfrak{f} with respect to the first variable; our aim is to study the equation $\partial_1 \mathfrak{f}(x, \lambda) = 0$. Observe that with our weak regularity assumptions on the group action (we do not assume in principle the differentiability of the map γ_g), it does not follow that if $\partial_1 \mathfrak{f}(x, \lambda) = 0$, then also $\partial_1 \mathfrak{f}(g \cdot x, \lambda) = 0$ for all $g \in G$. We will therefore explicitly assume:

- (C) for all $\lambda \in \Lambda$, the set $\{x \in \mathfrak{M} : \partial_1 \mathfrak{f}(x, \lambda) = 0\}$ is G -invariant.

A sufficient condition that guarantees (C) will be discussed below (see Lemma 3.1).

Let us now look at the question of lack of a *gradient* for the function \mathfrak{f} ; we define a *gradient-like* map by introducing a suitable vector bundle on the manifold \mathfrak{M} , defined by the following axioms:

- (D1) $\mathcal{E} \rightarrow \mathfrak{M}$ is a \mathcal{C}^k -Banach vector bundle;
 (D2) there exist \mathcal{C}^k -vector bundle morphisms:

$$i: T\mathfrak{M} \longrightarrow \mathcal{E} \quad \text{and} \quad j: \mathcal{E} \longrightarrow T\mathfrak{M}^*,$$

with j injective;

- (D3) for all $x \in \mathfrak{M}$, the bilinear form $\langle \cdot, \cdot \rangle_x: T_x \mathfrak{M} \times T_x \mathfrak{M} \rightarrow \mathbb{R}$ defined by $\langle u, v \rangle_x = j_x(i_x(u))v$ is a (not necessarily complete) positive definite inner product (this implies that also i is injective);
 (D4) there exists a \mathcal{C}^k -map $\delta \mathfrak{f}: \mathfrak{M} \times \Lambda \rightarrow \mathcal{E}$ such that

$$j \circ \delta \mathfrak{f} = \partial_1 \mathfrak{f}.$$

Since j is injective, the above gives that $\partial_1 \mathfrak{f}(x, \lambda) = 0$ if and only if $\delta \mathfrak{f}(x, \lambda) = 0$.

Let us now go back to the question of G -invariance of the set of critical points of the functions $\mathfrak{f}(\cdot, \lambda)$. Assuming that the G action is by diffeomorphisms (i.e., that the maps γ_g are diffeomorphisms), given x_0 such that $\partial_1 \mathfrak{f}(x_0, \lambda) = 0$ then obviously $\partial_1 \mathfrak{f}(g \cdot x_0, \lambda) = 0$ for all $g \in G$. For this conclusion it is necessary to differentiate γ_g ; when the action of G is only by homeomorphisms, the G -invariance of the critical set is obtained under a suitable assumption of G -equivariance for the map $\delta \mathfrak{f}$. Given $x \in \mathfrak{M}$, the fiber of \mathcal{E} over x will be denoted by \mathcal{E}_x . Consider the following:

- (E1) there exists a continuous left G -action by linear isomorphisms on the fibers of \mathcal{E} compatible with the action on \mathfrak{M} , i.e., such that the projection $\mathcal{E} \rightarrow \mathfrak{M}$ is equivariant (this means that for each g , it is given a family of linear isomorphisms $\varphi_{g,x}: \mathcal{E}_x \rightarrow \mathcal{E}_{g \cdot x}$ depending continuously on $x \in \mathfrak{M}$ and on $g \in G$, such that $\varphi_{gh,x} = \varphi_{g,h \cdot x} \circ \varphi_{h,x}$ for all $g, h \in G$ and all $x \in \mathfrak{M}$);
 (E2) the map $\delta \mathfrak{f}(\cdot, \lambda): \mathfrak{M} \rightarrow \mathcal{E}$ is equivariant for all $\lambda \in \Lambda$.

Lemma 3.1. *Axioms (E1) and (E2) imply (C).*

Proof. Assume $\partial_1 \mathfrak{f}(x_0, \lambda) = 0$, then $\delta \mathfrak{f}(x_0, \lambda) = 0$. The equivariance property gives $\delta \mathfrak{f}(g \cdot x_0, \lambda) = 0$ for all $g \in G$, i.e., $\partial_1 \mathfrak{f}(g \cdot x_0, \lambda) = 0$ for all $g \in G$. \square

Finally, another set of assumptions is needed in order to deal with the lack of the map $x \mapsto d\beta_x(1) \in \text{Lin}(\mathfrak{g}, T\mathfrak{M})$ for all $x \in \mathfrak{M}$. Our next set of hypotheses will give the existence of a continuous extension to \mathfrak{M} of this map provided that its

codomain be enlarged and endowed with a weaker topology. As above, this set of assumptions is better cast in terms of vector bundles and injective morphisms:

(F) there exists a \mathcal{C}^k -vector bundle $\mathcal{Y} \rightarrow \mathfrak{M}$ and \mathcal{C}^k -vector bundle morphisms:

$$\tilde{j}: \mathcal{E} \longrightarrow \mathcal{Y}^* \quad \text{and} \quad \kappa: T\mathfrak{M} \longrightarrow \mathcal{Y},$$

with κ injective, such that:

- (F1) $\kappa^* \circ \tilde{j} = j$ (from which it follows that also \tilde{j} is injective);
- (F2) the map $\mathfrak{M}' \ni x \mapsto \kappa_x \circ d\beta_x(1) \in \text{Lin}(\mathfrak{g}, \mathcal{Y}_x)$ has a continuous extension to a section of the vector bundle $\text{Lin}(\mathfrak{g}, \mathcal{Y}) \rightarrow \mathfrak{M}$.

From density of \mathfrak{M}' , the extension in (F2) is therefore unique.

We are now ready for:

Theorem 3.2. *In the above setup, let $(x_0, \lambda_0) \in \mathfrak{M}' \times \Lambda$ be a point such that:*

$$\partial_1 f(x_0, \lambda_0) = 0.$$

Denote by $L: T_{x_0}\mathfrak{M} \rightarrow \mathcal{E}_{x_0}$ the linear map

$$L := \pi_{\text{ver}} \circ \partial_1(\delta f)(x_0, \lambda_0),$$

where $\pi_{\text{ver}}: T_{0_{x_0}}\mathcal{E} \rightarrow \mathcal{E}_{x_0}$ is the canonical vertical projection. If:

- (G1) *L is Fredholm of index 0;*
- (G2) *$\ker L = \text{Im } d\beta_{x_0}(1)$,*

then there exists a G -invariant neighborhood $V \subset \mathfrak{M} \times \Lambda$ of $(G \cdot x_0, \lambda_0)$ and a \mathcal{C}^k -function $\sigma: \Lambda_0 \rightarrow \mathfrak{M}$ defined in a neighborhood Λ_0 of λ_0 in Λ such that $(x, \lambda) \in V$ and $\partial_1 f(x, \lambda) = 0$ hold if and only if $x \in G \cdot \sigma(\lambda)$.

Remark 3.3. Condition (G2) is an equivariant *nondegeneracy condition* on the critical orbit $G \cdot x_0$.

Proof. We will study a local problem first, and we will then use the group action for the proof of the global statement. After suitable local charts and local trivialization of vector bundles around the point (x_0, λ_0) have been chosen, one can assume the following situation:

- \mathfrak{M} is an open subset of a Banach space X , \mathfrak{M}' is a dense subset of \mathfrak{M} which is endowed with a topology finer than the induced topology from \mathfrak{M} , and Λ is an open subset of another Banach space;
- the group action on \mathfrak{M} is described by a map $\mathcal{U} \ni (g, x) \mapsto g \cdot x \in \mathfrak{M}$, with \mathcal{U} an open neighborhood of $\{1\} \times \mathfrak{M}$ in $G \times \mathfrak{M}$. Such map satisfies the obvious equalities given by group operations whenever⁵ such equalities make sense in the open set \mathcal{U} ;
- the \mathcal{C}^{k+1} -function $f: \mathfrak{M} \times \Lambda \rightarrow \mathbb{R}$ satisfies $f(g \cdot x, \lambda) = f(x, \lambda)$ wherever such equality makes sense (as above);
- the vector bundle \mathcal{E} is replaced with the product $\mathfrak{M} \times \mathcal{E}_0$, where \mathcal{E}_0 is a fixed Banach space (isometric to the typical fiber of \mathcal{E});
- $j: \mathfrak{M} \rightarrow \text{Lin}(\mathcal{E}_0, X^*)$ is a \mathcal{C}^k -map such that j_x is injective for all $x \in \mathfrak{M}$;

⁵ For instance, the equality $g \cdot (h \cdot x) = (gh) \cdot x$ holds for all $g, h \in G$ and $x \in \mathfrak{M}$ such that $(h, x) \in \mathcal{U}$ and $(g, h \cdot x) \in \mathcal{U}$. In particular, given $x \in \mathfrak{M}$, the equality must hold when g and h belong to some neighborhood of 1 in G . This could be formalized in terms of *partial actions* of groups (or groupoids) on topological spaces, but this is not relevant in the context of the present paper.

- $i: \mathfrak{M} \rightarrow \text{Lin}(X, \mathcal{E}_0)$ is a \mathcal{C}^k -map such that $j_x \circ i_x: X \rightarrow X^*$ is a (not necessarily complete) positive definite inner product on X (which implies in particular that i_x is injective for all x);
- the vector bundle \mathcal{Y} is replaced with the product $\mathfrak{M} \times \mathcal{Y}_0$, where \mathcal{Y}_0 is a fixed Banach space (isometric to the typical fiber of \mathcal{Y});
- $\tilde{j}: \mathfrak{M} \rightarrow \text{Lin}(\mathcal{E}_0, \mathcal{Y}_0^*)$ and $\kappa: \mathcal{M} \rightarrow \text{Lin}(X, \mathcal{Y}_0)$ are \mathcal{C}^k -maps taking values in the set of injective linear maps, and such that $\kappa_x^* \circ \tilde{j}_x = j_x$ for all $x \in \mathfrak{M}$;
- for all $x \in \mathfrak{M}$, the map β_x is only defined on an open neighborhood of 1 in G . For $x \in \mathfrak{M}'$, its derivative at 1 is a linear map $d\beta_x(1): \mathfrak{g} \rightarrow X$ that depends continuously on x , relatively to the finer topology of \mathfrak{M}' ;
- the map $\mathfrak{M}' \ni x \mapsto \kappa_x \circ [d\beta_x(1)] \in \text{Lin}(\mathfrak{g}, \mathcal{Y}_0)$ has a continuous extension to \mathfrak{M} ;
- $\partial_1 f: \mathfrak{M} \times \Lambda \rightarrow X^*$ and $\delta f: \mathfrak{M} \times \Lambda \rightarrow \mathcal{E}_0$ are maps of class \mathcal{C}^k such that $j_x(\delta f(x, \lambda)) = \partial_1 f(x, \lambda)$ for all (x, λ) ;
- the linear operator $L: X \rightarrow \mathcal{E}_0$ is given simply by the partial derivative $\partial_1(\delta f)(x_0, \lambda_0)$. It is a Fredholm operator of index zero, and $\ker L$ is given by the image of the linear map $d\beta_{x_0}(1)$.

Let $S = \text{Im}(d\beta_{x_0}(1))^\perp$ be the closed subspace of X given by the orthogonal complement of the subspace $\text{Im}(d\beta_{x_0}(1))$ relatively to the inner product $\langle \cdot, \cdot \rangle = j_{x_0} \circ i_{x_0}$. Since $\text{Im}(d\beta_{x_0}(1))$ is finite-dimensional, we have a direct sum decomposition

$$(3.2) \quad X = \text{Im}(d\beta_{x_0}(1)) \oplus S.$$

Let us now introduce a finite-dimensional subspace $Y \subset \mathcal{E}_0$ by:

$$Y = i_{x_0}(\ker L);$$

we claim that Y is complementary to the closed subspace $\text{Im } L$ in \mathcal{E}_0 . In order to prove the claim, we first observe that, using the fact that i_{x_0} is injective and L has index 0, then the dimension of Y equals the codimension of $\text{Im } L$. Thus, our claim is proved if we show that $Y \cap \text{Im } L = \{0\}$. We have a commutative diagram:

$$(3.3) \quad \begin{array}{ccc} X & \xrightarrow{L} & \mathcal{E}_0 \\ & \searrow \partial_1(\partial_1 f)(x_0, \lambda_0) & \downarrow j_{x_0} \\ & & X^* \end{array}$$

that is easily obtained differentiating the equality $j_x(\delta f(x, \lambda_0)) = \partial_1 f(x, \lambda_0)$ with respect to x at $x = x_0$, keeping in mind that $\delta f(x_0, \lambda_0) = 0$. Observe that the second line in (3.3) is a symmetric operator, and therefore we obtain:

$$(3.4) \quad j_{x_0}(\text{Im } L) \subset [\ker(j_{x_0} \circ L)]^\circ = (\ker L)^\circ,$$

where W° denotes the annihilator of the subspace $W \subset X$ in X^* . Now, if $v \in \ker L$ is such that $i_{x_0}(v) \in \text{Im } L$, then by (3.4), $j_{x_0} \circ i_{x_0}(v) \in (\ker L)^\circ$, and in particular $j_{x_0}(i_{x_0}(v))v = 0$. By assumption (D3), it follows that $v = 0$, i.e., $Y \cap \text{Im } L = \{0\}$, and therefore:

$$\mathcal{E}_0 = Y \oplus \text{Im } L.$$

Let $P: \mathcal{E}_0 \rightarrow \text{Im } L$ be the projection relative to this direct sum decomposition of \mathcal{E}_0 . We define the function:

$$\begin{aligned} H: (\mathfrak{M} \cap S) \times \Lambda &\longrightarrow \text{Im } L \\ H(x, \lambda) &= P(\delta f(x, \lambda)) \end{aligned}$$

observe that $H(x_0, \lambda_0) = 0$. Such map has the same regularity as δf . The derivative $\partial_1 H(x_0, \lambda_0)$ is $P \circ L|_S = L|_S: S \rightarrow \text{Im } L$, and this is an isomorphism by assumption (G2) and by (3.2). We can therefore apply the standard implicit function theorem to the equation $H(x, \lambda) = 0$ around (x_0, λ_0) , obtaining a neighborhood Λ_0 of λ_0 in Λ and a \mathcal{C}^k -function $\sigma: \Lambda_0 \rightarrow (\mathfrak{M} \cap S)$ with $\sigma(\lambda_0) = x_0$ and such that, given (x, λ) in a neighborhood of (x_0, λ_0) in $(\mathfrak{M} \cap S) \times \Lambda$, the equality $H(x, \lambda) = 0$ holds if and only if $x = \sigma(\lambda)$.

In order to complete the proof of our theorem, we will show the following facts:

- (1) there exists a neighborhood W of (x_0, λ_0) in $\mathfrak{M} \times \Lambda$ such that, given $(x, \lambda) \in W$, then $H(x, \lambda) = 0$ if and only if $\partial_1 f(x, \lambda) = 0$;
- (2) if $x \in \mathfrak{M}$ is sufficiently close to x_0 , then the orbit $G \cdot x$ has non-empty intersection with $\mathfrak{M} \cap S$.

By possibly reducing the domain of the function σ , we can assume that its graph is contained in W . The first claim implies that given (x, λ) sufficiently close to (x_0, λ_0) in $(\mathfrak{M} \cap S) \times \Lambda$, the equality $\partial_1 f(x, \lambda) = 0$ holds if and only if $x = \sigma(\lambda)$. The second claim and assumption (C) will imply that given (x, λ) sufficiently close to $G \cdot x_0 \times \{\lambda_0\}$ in $\mathfrak{M} \times \Lambda$, then $\partial_1 f(x, \lambda) = 0$ if and only if $x \in G \cdot \sigma(\lambda)$.

In order to prove claim (1), we observe first that if $(x, \lambda) \in (\mathfrak{M} \cap S) \times \Lambda$ and $\partial_1 f(x, \lambda) = 0$, then $\delta f(x, \lambda) = 0$ and therefore $H(x, \lambda) = 0$. Conversely, let us show that if $x \in \mathfrak{M} \cap S$ is near x_0 , and $H(x, \lambda) = 0$, then $\delta f(x, \lambda) = 0$ (and thus also $\partial_1 f(x, \lambda) = 0$). We observe that if $H(x, \lambda) = 0$, then $\delta f(x, \lambda) \in \mathfrak{i}_{x_0}(\ker L)$, and thus, $\tilde{j}_x(\delta f(x, \lambda))$ annihilates $[\tilde{j}_x(\mathfrak{i}_{x_0}(\ker L))]_o$. Here, given a subspace $Z \subset X^*$, the symbol Z_o denotes the subspace of X annihilated by Z . Denote by $B: \mathfrak{M} \rightarrow \text{Lin}(\mathfrak{g}, \mathcal{Y}_0)$ the continuous extension of the map $x \mapsto \kappa_x \circ [d\beta_x(1)]$ defined in \mathfrak{M}' (by (F2)); we claim that $\tilde{j}_x(\delta f(x, \lambda))$ annihilates also the image of $B(x)$. This follows from the fact that for $x \in \mathfrak{M}'$, $\partial_1 f(x, \lambda)$ annihilates $\text{Im}(d\beta_x(1))$, which is easily seen by differentiating at $g = 1$ the (constant) map $g \mapsto f(\beta_x(g), \lambda)$ (use (B1)), and a continuity argument. Namely, observe that for $x \in \mathfrak{M}'$:

$$\begin{aligned} 0 &= \partial_1 f(x, \lambda) \circ d\beta_x(1) = j_x(\delta f(x, \lambda)) \circ [d\beta_x(1)] \\ &= \kappa_x^*(\tilde{j}_x(\delta f(x, \lambda))) \circ [d\beta_x(1)] = \tilde{j}_x(\delta f(x, \lambda)) \circ \kappa_x \circ [d\beta_x(1)]. \end{aligned}$$

This says that the map:

$$\mathfrak{M} \ni x \longmapsto \tilde{j}_x(\delta f(x, \lambda)) \circ B(x) \in \mathfrak{g}^*$$

vanishes for $x \in \mathfrak{M}'$. Thus, by continuity, it vanishes identically, i.e., $\tilde{j}_x(\delta f(x, \lambda))$ annihilates the image of $B(x)$.

To conclude the proof of (1), it suffices to show that for $x \in \mathfrak{M}$ near x_0 one has:

$$(3.5) \quad \text{Im}(B(x)) + [\tilde{j}_x(\mathfrak{i}_{x_0}(\ker L))]_o = \mathcal{Y}_0.$$

Using the continuity of B and the fact that the subspace $[\tilde{j}_x(\mathfrak{i}_{x_0}(\ker L))]_o$ has *fixed* codimension in \mathcal{Y}_0 (i.e., it does not depend on x), it follows that condition (3.5) is

open⁶ in \mathfrak{M} . Thus, it suffices to show that it holds at $x = x_0$. Let us check equality (3.5) at $x = x_0$. The dimension of $\text{Im}(B(x_0))$ equals the dimension of $\text{Im}(d\beta_{x_0}(1))$, and this is equal to the dimension of $\ker L$, by assumption (G2). Since \tilde{j}_{x_0} and i_{x_0} are injective, then the codimension of $[\tilde{j}_{x_0}(i_{x_0}(\ker L))]_o$ is equal to the dimension of $\ker L$. Thus, it suffices to show:

$$\kappa_{x_0}(\text{Im}(d\beta_{x_0}(1))) \cap [\tilde{j}_{x_0}(i_{x_0}(\ker L))]_o = \{0\},$$

i.e.,

$$\begin{aligned} \text{Im}(d\beta_{x_0}(1)) \cap \kappa_{x_0}^{-1}[\tilde{j}_{x_0}(i_{x_0}(\ker L))]_o &= \ker L \cap [\kappa_{x_0}^* \circ \tilde{j}_{x_0}(i_{x_0}(\ker L))]_o \\ &= \ker L \cap [j_{x_0}(i_{x_0}(\ker L))]_o = \ker L \cap (\ker L)^\perp = \{0\}. \end{aligned}$$

It remains to show claim (2), i.e., equivalently, that the set $G \cdot S$ contains an open neighborhood of x_0 . Since we are not assuming differentiability of the group action, this does not follow from a transversality argument. The correct argument in the continuous case uses the notion of topological degree of a map, and it will be given separately in Proposition 3.4. In our case, this result is used setting $A = \mathfrak{M}$, $M = G$, $N = \mathfrak{M}$, $P = S$, $m_0 = 1$, $a_0 = x_0$ and χ is the action. \square

Proposition 3.4. *Let M be a finite-dimensional manifold, N a (possibly infinite-dimensional) Banach manifold, $P \subset N$ a Banach submanifold, and A a topological space. Assume that $\chi: A \times M \rightarrow N$ is a continuous function such that there exists $a_0 \in A$ and $m_0 \in M$ with:*

- (a) $\chi(a_0, m_0) \in P$;
- (b) $\chi(a_0, \cdot): M \rightarrow N$ is of class \mathcal{C}^1 ;
- (c) $\partial_2 \chi(a_0, m_0)(T_{m_0}M) + T_{\chi(a_0, m_0)}P = T_{\chi(a_0, m_0)}N$.

Then, for $a \in A$ near a_0 , $\chi(a, M) \cap P \neq \emptyset$.

Proof. Given a function $f: U \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$ of class \mathcal{C}^1 , where U is an open neighborhood of 0, such that $f(0) = 0$ and $df(0)$ an isomorphism, then the induced map $\tilde{f}: \mathbb{S}^{d-1} \rightarrow \mathbb{S}^{d-1}$ has topological degree equal to ± 1 . Here, \tilde{f} is defined by $\tilde{f}(x) = \|f(xr)\|^{-1}f(xr)$, where $r > 0$ is such that 0 is the unique zero of f in the closed ball $B[0; r]$ of \mathbb{R}^d .

Now, if A is any topological space, $f: A \times U \rightarrow \mathbb{R}^d$ is continuous, and $a_0 \in A$ is such that $f(a_0, \cdot)$ is of class \mathcal{C}^1 , $f(a_0, 0) = 0$ and $\partial_2 f(a_0, 0)$ is an isomorphism, for a near a_0 , and $r > 0$ sufficiently small, $0 \in f(a, B[0; r])$. This follows from the continuity of the topological degree. The same holds for a function $f: A \times U \rightarrow \mathbb{R}^d$, where now U is an open neighborhood of 0 in \mathbb{R}^s , with $s \geq d$, under the assumption that $f(a_0, \cdot)$ be of class \mathcal{C}^1 , $f(a_0, 0) = 0$, and $\partial_2 f(a_0, 0)$ be surjective. Namely, it suffices to apply the argument above to the function obtained by restricting f to a d -dimensional subspace where $\partial_2 f(a_0, 0)$ is an isomorphism.

To prove the result, use local coordinates adapted to P in N , and assume that M , P and N are Banach spaces, with $N = P \oplus \mathbb{R}^d$, $d \leq s = \dim(M)$ is the codimension of P , and $m_0 = 0$. In this situation, the result is obtained applying the argument above to the function $f: A \times M \rightarrow \mathbb{R}^d$ given by $f(a, m) = \pi(\chi(a, m))$, where

⁶Details on the proof of openness of condition (3.5) are as follows. Let e_1, \dots, e_r be a basis of $\ker L$; the covectors $\omega_i = \tilde{j}_x(i_{x_0}(e_i))$, $i = 1, \dots, r$, are linearly independent in \mathcal{Y}_0^* . Consider the surjective linear map $\tau_x: \mathcal{Y}_0 \rightarrow \mathbb{R}^r$ defined by $\tau_x(v) = (\omega_1(v), \dots, \omega_r(v))$. The map $\mathfrak{M} \ni x \mapsto \tau_x \in \text{Lin}(\mathcal{Y}_0, \mathbb{R}^r)$ is continuous. Condition (3.5) is equivalent to $\text{Im}(B(x)) + \ker \tau_x = \mathcal{Y}_0$, i.e., that the linear map $\tau_x \circ B(x): \mathfrak{g} \rightarrow \mathbb{R}^r$ be surjective. This is clearly an open condition.

$\pi: N \rightarrow \mathbb{R}^d$ is the projection relative to the decomposition $N = P \oplus \mathbb{R}^d$. Clearly, $f(a, m) = 0$ if and only if $\chi(a, m) \in P$. Assumption (a) implies that $f(a_0, 0) = 0$, and assumption (c) implies that $\partial_2 f(a_0, 0)$ is surjective. \square

Remark 3.5. In Theorem 3.2, some assumptions on the group action can be weakened. For instance, the result holds also for *local* group actions, see Appendix A. This version of the equivariant implicit function theorem will be used in the constant mean curvature problem, Subsection 4.1. Versions of the result for the so-called *partial actions* of groups, or even for actions of groupoids, semigroups, monoids, etc., are also possible.

4. APPLICATIONS TO GEOMETRIC VARIATIONAL PROBLEMS

We now describe concrete applications (Propositions 4.1, 4.4 and 4.5) of our abstract result, Theorem 3.2, to three classic geometric variational problems: constant mean curvature hypersurfaces, closed geodesics and harmonic maps.

4.1. CMC hypersurfaces. Let us consider the CMC hypersurfaces variational problem, studied in Section 2, as a first example that fits into the variational framework described by the axioms (A)–(G). We will obtain an improvement of Proposition 2.3 using a version of Theorem 3.2 for local actions, see Theorem A.2. We need a technical assumption concerning the existence of an *invariant volume functional* around a given CMC embedding $x: M \hookrightarrow \overline{M}$. This will be a volume functional invariant under left-compositions with isometries of the ambient space, see Definition B.1. Examples where this assumption is satisfied are discussed in Appendix B. We stress that this assumption is indeed necessary, see Example 1. Henceforth in this subsection, assume that M is a compact manifold, and that $(\overline{M}, \overline{g})$ is a Riemannian manifold, with $\dim(\overline{M}) = \dim(M) + 1$.

Proposition 4.1. *Let $x: M \hookrightarrow \overline{M}$ be a nondegenerate and transversely oriented CMC embedding, with mean curvature⁷ H_0 . Assume that there exists an invariant volume functional \mathcal{V} defined in a neighborhood of x in the set of \mathcal{C}^1 -embeddings of M into \overline{M} . Then, there exists an open interval $]H_0 - \varepsilon, H_0 + \varepsilon[$ and a smooth function $]H_0 - \varepsilon, H_0 + \varepsilon[\ni H \mapsto \varphi_H \in \mathcal{C}^{2,\alpha}(M)$, with $\varphi_{H_0} = 0$, such that:*

- (a) *for all $H \in]H_0 - \varepsilon, H_0 + \varepsilon[$, the map $x_H: M \hookrightarrow \overline{M}$ defined by*

$$x_H(p) = \exp_{x(p)}(\varphi_H(p) \cdot \vec{n}_x(p)), \quad p \in M,$$

is a CMC embedding having mean curvature equal to H ;

- (b) *any given CMC embedding $y: M \hookrightarrow \overline{M}$ sufficiently close to x (in the $\mathcal{C}^{2,\alpha}$ -topology) is isometrically congruent to some x_H .*

Proof. We will use the following variational setup. Consider the set $\mathfrak{E}(M, \overline{M})$ of all *unparameterized embeddings* of class $\mathcal{C}^{2,\alpha}$ of M into \overline{M} , i.e., the set of congruence classes of $\mathcal{C}^{2,\alpha}$ embeddings $y: M \hookrightarrow \overline{M}$. Such a set does not have a natural global differentiable structure, but it admits an atlas of charts that make it an infinite-dimensional topological manifold modeled on the Banach space $\mathcal{C}^{2,\alpha}(M)$, see [1]. Given a smooth embedding $y: M \rightarrow \overline{M}$, nearby congruence classes of embeddings are parameterized by sections of the normal bundle of y , using the exponential map of $(\overline{M}, \overline{g})$. We will identify congruence classes of embeddings near x with

⁷If $H_0 \neq 0$, then the assumption of transverse orientation is automatically satisfied. Namely, one can choose the transverse orientation given by the mean curvature vector.

functions belonging to a neighborhood of 0 in the Banach space $\mathcal{C}^{2,\alpha}(M)$; for this identification the transversal orientation of $x(M)$ is used. Let \mathfrak{M} be a sufficiently small neighborhood of x in $\mathfrak{E}(M, \overline{M})$, identified with a neighborhood of 0 in the space $\mathcal{C}^{2,\alpha}(M)$.

Consider the isometry group $G = \text{Iso}(\overline{M}, \overline{\mathfrak{g}})$ of the ambient manifold. There is a *local action* (see Appendix A) of G on \mathfrak{M} , defined as follows. If $y: M \hookrightarrow \overline{M}$ is an embedding near x and ϕ is an isometry of $(\overline{M}, \overline{\mathfrak{g}})$, then the action of ϕ on (the congruence class of) y is given by the (congruence class of the) left-composition $\phi \circ y$. The domain of this action consists of pairs (ϕ, y) such that $\phi \circ y$ belongs to \mathfrak{M} ; the axioms of local actions are readily verified for this map. The local action of G on the set of unparameterized embedding is continuous (see [1]), but the action is differentiable only on the dense subset \mathfrak{M}' of \mathfrak{M} consisting of congruence classes of embeddings of class $\mathcal{C}^{3,\alpha}$. The orbit of each of these elements is a \mathcal{C}^1 -submanifold of $\mathcal{C}^{2,\alpha}(M)$.

Given an embedding $y: M \hookrightarrow \overline{M}$, denote by $\mathcal{A}(y)$ the volume of M relatively to the volume form of the pull-back metric $y^*(\overline{\mathfrak{g}})$, and consider an invariant volume functional \mathcal{V} defined in a neighborhood of x in $\mathcal{C}^1(M, \overline{M})$. The values $\mathcal{A}(y)$ and $\mathcal{V}(y)$ do not depend on the parameterization of y , and \mathcal{A} and \mathcal{V} define functions on \mathfrak{M} that are smooth in every local chart, see [1] for details. Finally, let Λ be an open interval of \mathbb{R} containing H_0 , and denote by $\mathfrak{f}: \mathfrak{M} \times \Lambda \rightarrow \mathbb{R}$ the function $\mathfrak{f}(y, \lambda) = \mathcal{A}(y) + \frac{\lambda}{m} \cdot \mathcal{V}(y)$. It is well known (see [2, 3]) that $\partial_1 \mathfrak{f}(y, \lambda) = 0$ if and only if y is a CMC embedding with mean curvature equal to λ . The second variation of $\mathfrak{f}(\cdot, H_0)$ at x is identified with the Jacobi operator J_x in (2.5). In particular, $J_x: \mathcal{C}^{2,\alpha}(M) \rightarrow \mathcal{C}^{0,\alpha}(M)$ is a Fredholm operator of index 0, see [25, Thm 1.1, (2)].

Since \mathcal{A} and \mathcal{V} are invariant under the local action of left-composition with elements of the isometry group $G = \text{Iso}(\overline{M}, \overline{\mathfrak{g}})$, then so is the function \mathfrak{f} . Note that G is a Lie group, see [15], which is compact when \overline{M} is compact.

The result is proved as a direct application of the equivariant implicit function theorem for local actions, see Theorem A.2; the objects described in the axioms (A), (B), (D) and (F) are defined as follows for the CMC variational problem.

- \mathcal{E} is the Banach space $\mathcal{C}^{0,\alpha}(M)$;
- \mathfrak{i} is the inclusion $\mathcal{C}^{2,\alpha} \hookrightarrow \mathcal{C}^{0,\alpha}$ and \mathfrak{j} is induced by the L^2 -pairing $(f, g) \mapsto \int_M f \cdot g \, \text{vol}_{\overline{\mathfrak{g}}}$;
- \mathcal{Y} is the Banach space $\mathcal{C}^{1,\alpha}(M)$;
- $\tilde{\mathfrak{j}}$ is induced by the L^2 -pairing;
- κ is the inclusion;
- identifying the Lie algebra \mathfrak{g} with the space of (complete) Killing vector fields on $(\overline{M}, \overline{\mathfrak{g}})$, for $y \in \mathfrak{M}'$, the map $d\beta_y(1): \mathfrak{g} \rightarrow T_y \mathfrak{M}$ associates to a Killing vector field \overline{K} the orthogonal component of \overline{K} along y ;
- given a $\mathcal{C}^{2,\alpha}$ -embedding y , $\delta \mathfrak{f}(y, \lambda)$ is the mean curvature function of y (which is a $\mathcal{C}^{0,\alpha}$ -function on M);
- $\partial_1(\delta \mathfrak{f})(x, H_0)$ is identified with the Jacobi operator J_x .

The assumptions of Theorem 3.2/Theorem A.2 are easily verified, and the conclusion follows. \square

The following examples show that neither the assumption on the existence of an invariant volume functional nor the assumption on the transversal orientation in the case of minimal embeddings can be omitted in Proposition 4.1.

Example 1. Consider $M = \mathbb{S}^1$ and $\overline{M} = \mathbb{S}^1 \times \mathbb{S}^1$ is the 2-torus endowed with the flat metric. The embedding $x: \mathbb{S}^1 \rightarrow \mathbb{S}^1 \times \mathbb{S}^1$ given by $x(z) = (z, 1)$, $z \in \mathbb{S}^1$, is obviously minimal (i.e., a geodesic). It is also easy to see that such embedding is nondegenerate, i.e., every periodic Jacobi field along x is the restriction of a Killing vector field. However, near x there exists no embedding of \mathbb{S}^1 into $\mathbb{S}^1 \times \mathbb{S}^1$ with constant geodesic curvature different from zero. Namely, every constant geodesic curvature embedding should be the projection on $\mathbb{S}^1 \times \mathbb{S}^1$ of a circle in the plane \mathbb{R}^2 ; such a projection is a curve with trivial homotopy class, hence it cannot be close to x in the \mathcal{C}^1 -topology. Observe that, in this example, the image of x is not contained in any open subset of $\mathbb{S}^1 \times \mathbb{S}^1$ with trivial first cohomology space, and there exists no volume functional defined in a neighborhood of x in $\mathcal{C}^1(\mathbb{S}^1, \mathbb{S}^1 \times \mathbb{S}^1)$ which is invariant under isometries of $\mathbb{S}^1 \times \mathbb{S}^1$.

Example 2. We observe that the transverse orientability is a necessary condition in Proposition 4.1. Namely, such condition is closed (and also open) relatively to the \mathcal{C}^1 -topology; thus, if $(x_H)_{H \in]-\varepsilon, \varepsilon[}$ is a continuous family of CMC embeddings, such that each x_H has mean curvature H , then x_0 must be transversely oriented.

For instance, consider the real projective plane \mathbb{RP}^2 with the standard metric, and the minimal (i.e., geodesic) embedding $x_0: \mathbb{S}^1 \hookrightarrow \mathbb{RP}^2$ obtained by projecting in \mathbb{RP}^2 a minimal geodesic between two antipodal points in \mathbb{S}^2 . This is a nondegenerate minimal embedding, which is not transversely oriented. The only CMC immersions of \mathbb{S}^1 in \mathbb{S}^2 are parallel to the equator. The corresponding immersions obtained in the quotient $x_H: \mathbb{S}^1 \hookrightarrow \mathbb{RP}^2$ are not \mathcal{C}^1 -close to x_0 since, for instance, the length of x_h tends to twice the length of x_0 as H goes to 0.

An alternative form of stating Proposition 4.1 uses the notion of *rigidity* for a path of CMC embeddings.

Definition 4.2. Given a 1-parameter family $x_s: M \hookrightarrow \overline{M}$, $s \in [a, b]$, of CMC embeddings, we say that the family $X = \{x_s\}_{s \in [a, b]}$ is *rigid* if there exists an open neighborhood \mathcal{U} of X in $\mathcal{C}^{2, \alpha}(M, \overline{M})$ such that any CMC embedding $x: M \hookrightarrow \overline{M}$ in \mathcal{U} is isometrically congruent to some x_s . We say that the family is *locally rigid* at $s_0 \in [a, b]$ if there exists $\varepsilon > 0$ such that, setting $I = [a, b] \cap [s_0 - \varepsilon, s_0 + \varepsilon]$, the family $\{x_s\}_{s \in I}$ is rigid.

Corollary 4.3. Let $x_s: M \hookrightarrow \overline{M}$, $s \in [a, b]$, be a \mathcal{C}^1 -family of CMC embeddings, denote by $\mathcal{H}(s)$ the mean curvature of x_s , and let $s_0 \in [a, b]$ be such that:

- x_{s_0} is nondegenerate;
- there exists an invariant volume functional in a \mathcal{C}^1 -neighborhood of x_{s_0} ;
- $\mathcal{H}'(s_0) \neq 0$.

Then, $X = \{x_s\}_{s \in [a, b]}$ is locally rigid at s_0 .

Proof. The assumption $\mathcal{H}'(s_0) \neq 0$ implies the existence of a \mathcal{C}^1 -function

$$]\mathcal{H}(s_0) - \varepsilon, \mathcal{H}(s_0) + \varepsilon[\ni H \mapsto s(H) \in]s_0 - \delta, s_0 + \delta[,$$

with $\varepsilon, \delta > 0$ small enough, such that $\mathcal{H}(s(H)) = H$ for all H . Apply Proposition 4.1 to $x = x_{s_0}$, obtaining a new path $H \mapsto x_H$ of CMC embeddings. Note that x_{s_0} must be transversely oriented, even in the case $\mathcal{H}(s_0) = 0$; namely, by the assumption $\mathcal{H}'(s_0) \neq 0$, it follows that \mathcal{H} is not constant in any neighborhood of s_0 , and thus x_{s_0} is limit of transversely oriented embeddings. By part (b) of

Proposition 4.1, $x_{s(H)}$ must be isometrically congruent to X_H for all H near $H(s_0)$, and the local rigidity follows readily. \square

The case of CMC embeddings of manifolds with boundary. A result totally analogous to that of Proposition 4.1 holds also in the case of codimension one CMC embeddings $x: M \hookrightarrow \overline{M}$ of manifolds M with boundary ∂M . In this situation, one is interested in variations of x that fix the boundary, and the corresponding infinitesimal variations are Jacobi fields that vanish on ∂M . Thus, a CMC embedding $x: M \hookrightarrow \overline{M}$ is *nondegenerate* if every Jacobi field f along x that vanishes on ∂M is of the form $f = \overline{\mathbf{g}}(\overline{K}, \vec{n}_x)$ for some Killing field \overline{K} of $(\overline{M}, \overline{\mathbf{g}})$ tangent to $x(\partial M)$. If x is nondegenerate in this sense, then the implicit function theorem gives the existence of a variation $(x_H)_{H \in]H_0 - \varepsilon, H_0 + \varepsilon[}$ of x by CMC embeddings $x_H: M \hookrightarrow \overline{M}$ such that $x_H|_{\partial M} = x|_{\partial M}$ for all H .

The proof of the fixed boundary CMC implicit function theorem is totally analogous to that of Proposition 4.1, *mutatis mutandis*. It is required the existence of a volume functional defined in a \mathcal{C}^1 -neighborhood of x in the set of embeddings $y: M \hookrightarrow \overline{M}$ with fixed boundary, i.e., $y(\partial M) = x(\partial M)$, and which is invariant under isometries of $(\overline{M}, \overline{\mathbf{g}})$ that preserve $x(\partial M)$. Note that the group of such isometries is always compact, because the action of the isometry group is proper and ∂M is compact. In this case, invariant volume functionals can be obtained from invariant primitives of the volume form, using an averaging procedure, see Appendix B.

As to the variational framework, now \mathfrak{M} is the manifold of fixed boundary unparameterized embeddings of M into \overline{M} of class $\mathcal{C}^{2,\alpha}$, which is modeled on the Banach space $\mathcal{C}_0^{2,\alpha}(M, \mathbb{R}) = \{f \in \mathcal{C}^{2,\alpha}(M, \mathbb{R}) : f|_{\partial M} \equiv 0\}$. Note that, when M has boundary, the Jacobi operator $J_x: \mathcal{C}_0^{2,\alpha}(M, \mathbb{R}) \rightarrow \mathcal{C}^0(M, \mathbb{R})$ is Fredholm of index 0.

Natural extensions of the CMC implicit function theorem. The result of Proposition 4.1 extends naturally to more general situations involving hypersurfaces that are stationary for a parametric elliptic functional with a volume constraint, like for instance hypersurfaces with constant *anisotropic mean curvature*, see [17]. Such extension is quite straightforward, and will not be discussed here. It is also interesting to point out that Proposition 4.1 can be extended to the case of CMC *immersions*, rather than embeddings. The question here is to endow the set of unparameterized immersions with a local differential structure based on the exponential map of the normal bundle. This is possible in the case of the so-called *free immersions*, i.e., immersions $x: M \rightarrow \overline{M}$ with the property that the unique diffeomorphism ϕ of M satisfying $x \circ \phi = x$ is the identity. This is the case, for instance, when there is some point in the image of x whose inverse image consists of a single point of M . Details can be found in [7].

4.2. Closed geodesics. As a second geometric application of Theorem 3.2, we study the closed geodesic variational problem on (pseudo-)Riemannian manifolds.

Let M be an arbitrary manifold, let \mathfrak{M} be the Banach manifold $\mathcal{C}^2(\mathbb{S}^1, M)$ consisting of all closed curves of class \mathcal{C}^2 in M , let \mathcal{B} be a Banach space of symmetric $(0, 2)$ -tensors of class \mathcal{C}^k on M , with $k \geq 3$, and let Λ denote an open subset of \mathcal{B} consisting of tensors that are everywhere nondegenerate on M . Thus, elements of Λ are (pseudo-)Riemannian metric tensors on M . We also fix an auxiliary Riemannian metric \mathbf{g}_R on M ; this metric induces a positive definite inner product and a norm $\|\cdot\|_R$ on each tangent and cotangent space to M , and on all tensor

products of these spaces. This will be used implicitly throughout whenever our constructions require the use of a norm or of an inner product of tensors.⁸ Given a (pseudo-)Riemannian metric tensor \mathbf{g} on M , let us denote by $T_{\mathbf{g}}$ the \mathbf{g}_R -symmetric $(1, 1)$ -tensor on M defined by:

$$(4.1) \quad \mathbf{g} = \mathbf{g}_R(T_{\mathbf{g}} \cdot, \cdot).$$

Consider the smooth function $\mathfrak{f}: \mathfrak{M} \times \Lambda \rightarrow \mathbb{R}$ given by:

$$\mathfrak{f}(\gamma, \mathbf{g}) = \frac{1}{2} \int_{\mathbb{S}^1} \mathbf{g}(\gamma', \gamma') d\theta;$$

for a given $\mathbf{g}_0 \in \Lambda$, the critical points of the map $\gamma \mapsto \mathfrak{f}(\gamma, \mathbf{g}_0)$ are precisely the periodic \mathbf{g}_0 -geodesics on M . For $\gamma \in \mathfrak{M}$, the tangent space $T_{\gamma}\mathfrak{M}$ is identified with the Banach space of all periodic vector fields V of class \mathcal{C}^2 along γ . Given such a $V \in T_{\gamma}\mathfrak{M}$, the derivative $\partial_1 \mathfrak{f}(\gamma, \mathbf{g})V$ is given by:

$$(4.2) \quad \partial_1 \mathfrak{f}(\gamma, \mathbf{g})V = \int_{\mathbb{S}^1} \mathbf{g}(\gamma', \frac{D\mathbf{g}}{d\theta} V) d\theta,$$

where $\frac{D\mathbf{g}}{d\theta}$ is the covariant derivative operator along γ relative to the Levi-Civita connection $\nabla^{\mathbf{g}}$ of \mathbf{g} . Recall that if γ is a \mathbf{g} -geodesic, the *Jacobi operator* J along γ is the linear differential operator:

$$(4.3) \quad J(V) = \left(\frac{D\mathbf{g}}{d\theta^2} \right)^2 V + R^{\mathbf{g}}(\gamma', V)\gamma',$$

defined in the space of \mathcal{C}^2 -vector fields V along γ . Here $R^{\mathbf{g}}$ is the curvature tensor⁹ of the Levi-Civita connection of \mathbf{g} . A Jacobi field along γ is a vector field V satisfying $J(V) = 0$. A closed \mathbf{g} -geodesic γ is said to be *nondegenerate* if the only periodic Jacobi fields along γ are (constant) multiples of the tangent field γ' . Nondegeneracy of all closed geodesics (including iterates) is a *generic* property in the set of (pseudo-)Riemannian metric tensors \mathbf{g} , see [4, 5] for a precise statement.

Let G be the circle \mathbb{S}^1 , acting on \mathfrak{M} by rotation on the parameter, i.e., by right-composition. This action is only continuous (and not differentiable), but each $g \in G$ gives a diffeomorphism of \mathfrak{M} . The stabilizer of every non-constant closed curve γ in M is a finite cyclic subgroup of \mathbb{S}^1 . When such stabilizer is trivial, we say that γ is *prime*, i.e., it is not the iterate of some other closed curve in M . If $n > 1$ is the order of the stabilizer of a curve γ , then γ is the n -fold iterate of some prime closed curve on M . Two closed curves γ_1 and γ_2 on M belong to the same \mathbb{S}^1 -orbit if and only if:

- (a) $\gamma_1(\mathbb{S}^1) = \gamma_2(\mathbb{S}^1)$;
- (b) γ_1 and γ_2 have stabilizers of the same order.

When (a) and (b) are satisfied, we will say that γ_1 and γ_2 are *geometrically equivalent*.

⁸These norms can be used, e.g., to give a simple construction of the Banach space \mathcal{B} . Consider ∇^R the Levi-Civita connection of \mathbf{g}_R . Then \mathcal{B} may be taken as the space of $(0, 2)$ -tensors s of class \mathcal{C}^k on M that are \mathbf{g}_R -bounded, i.e., such that

$$\|s\|_{\mathcal{B}} = \max_{1 \leq i \leq k} \left\{ \sup_{x \in M} \left\| (\nabla^R)^i s(x) \right\|_R \right\} < +\infty.$$

⁹The sign convention adopted for the curvature tensor is $R^{\mathbf{g}}(X, Y) = [\nabla_X^{\mathbf{g}}, \nabla_Y^{\mathbf{g}}] - \nabla_{[X, Y]}^{\mathbf{g}}$.

In this context, we may state the following consequence of the equivariant implicit function theorem.

Proposition 4.4. *Let \mathbf{g} be a \mathcal{C}^k (pseudo-)Riemannian metric tensor on the manifold M , with $k \geq 3$, and let γ be a closed \mathbf{g} -geodesic on M . Consider Λ an open subset of a Banach space of symmetric $(0, 2)$ -tensors of class \mathcal{C}^k on M , with $\mathbf{g} \in \Lambda$, such that every tensor in Λ is a (pseudo-)Riemannian metric tensor on M . Suppose γ is nondegenerate, i.e., all periodic Jacobi fields along γ are (constant) multiples of the tangent field γ' . Then there exists a neighborhood \mathcal{U} of \mathbf{g} in Λ , a neighborhood \mathcal{V} of γ in $\mathcal{C}^2(\mathbb{S}^1, M)$, and a \mathcal{C}^{k-1} function $\mathcal{U} \ni \mathbf{h} \mapsto \gamma_{\mathbf{h}} \in \mathcal{C}^2(\mathbb{S}^1, M)$ such that:*

- (a) $\gamma_{\mathbf{h}}$ is a closed \mathbf{h} -geodesic in M for all $\mathbf{h} \in \mathcal{U}$;
- (b) if $\mathbf{h} \in \mathcal{U}$ and $\alpha \in \mathcal{V}$ is a closed \mathbf{h} -geodesic in M , then α is geometrically equivalent to $\gamma_{\mathbf{h}}$.

Proof. Let us describe how to verify that all assumptions of Theorem 3.2 are satisfied by the closed geodesic problem, proving the above result. Consider the following objects:

- \mathfrak{M}' is the set $\mathcal{C}^3(\mathbb{S}^1, M)$, endowed with the \mathcal{C}^3 -topology;
- \mathcal{E} is the mixed vector bundle whose fiber \mathcal{E}_{γ} is the Banach space of all periodic continuous vector fields along γ , endowed with the topology \mathcal{C}^2 on the base and \mathcal{C}^0 on the fibers;
- \mathbf{i} is the inclusion, and \mathbf{j} is induced by the L^2 -pairing (this uses the inner product given by \mathbf{g}_R);
- \mathcal{Y} is the mixed vector bundle whose fiber \mathcal{Y}_{γ} is the Banach space of all periodic \mathcal{C}^1 -vector fields along γ , endowed with the topology \mathcal{C}^2 on the base and \mathcal{C}^1 on the fibers;
- $\tilde{\mathbf{j}}$ is induced by the L^2 -pairing (this uses the inner product given by \mathbf{g}_R);
- κ is the inclusion;
- using the identification $\mathfrak{g} \cong \mathbb{R}$, $\text{Lin}(\mathfrak{g}, T\mathfrak{M}) \cong T\mathfrak{M}$ and $\text{Lin}(\mathfrak{g}, \mathcal{Y}) \cong \mathcal{Y}$, for $\gamma \in \mathfrak{M}'$, the map $\text{d}\beta_{\gamma}(1)$ is the element $\gamma' \in T\mathfrak{M}$;
- the map $\kappa \circ [\text{d}\beta_{\gamma}(1)]$ has the same expression of $\text{d}\beta_{\gamma}(1)$, where now $\gamma \in \mathfrak{M}$ and $\gamma' \in \mathcal{Y}$;
- the map $\delta\mathbf{f}$ is defined by $\delta\mathbf{f}(\gamma, \mathbf{g}) = -\mathbf{T}_{\mathbf{g}}(\frac{\text{D}\mathbf{g}}{\text{d}\theta}\gamma')$, where $\mathbf{T}_{\mathbf{g}}$ is defined in (4.1). Note that $\frac{\text{D}\mathbf{g}}{\text{d}\theta}\gamma'$ is a continuous vector field along γ , and:

$$\begin{aligned} \mathbf{j}_{\gamma}(\delta\mathbf{f}(\gamma, \mathbf{g}))V &= \int_{\mathbb{S}^1} \mathbf{g}_R(\delta\mathbf{f}(\gamma, \mathbf{g}), V) \text{d}\theta = - \int_{\mathbb{S}^1} \mathbf{g}_R(\mathbf{T}_{\mathbf{g}}(\frac{\text{D}\mathbf{g}}{\text{d}\theta}\gamma'), V) \text{d}\theta \\ &= - \int_{\mathbb{S}^1} \mathbf{g}(\frac{\text{D}\mathbf{g}}{\text{d}\theta}\gamma', V) \text{d}\theta = \int_{\mathbb{S}^1} \mathbf{g}(\gamma', \frac{\text{D}\mathbf{g}}{\text{d}\theta}V) \text{d}\theta = \partial_1\mathbf{f}(\gamma, \mathbf{g})V \end{aligned}$$

- The derivative $\partial_1(\delta\mathbf{f})$ is given by:

$$\partial_1(\delta\mathbf{f})(\gamma, \mathbf{g})V = -(\nabla_V^{\mathbf{g}} \mathbf{T}_{\mathbf{g}})(\frac{\text{D}\mathbf{g}}{\text{d}\theta}\gamma') - \mathbf{T}_{\mathbf{g}}(J(V)),$$

where $V \in T_{\gamma}\mathfrak{M}$ and J is the Jacobi operator (4.3).

The operator J acting on the space of periodic fields of class \mathcal{C}^2 along γ and taking values in the space of periodic continuous vector fields along γ is a Fredholm operator of index 0, as it is a compact perturbation of an isomorphism. Since the composition on the left with $\mathbf{T}_{\mathbf{g}}$ is an isomorphism, it follows that the operator $V \mapsto \mathbf{T}_{\mathbf{g}}(J(V))$ is a Fredholm operator of index 0 from the space of periodic fields of class \mathcal{C}^2 along γ to the space of periodic

continuous vector fields along γ . The operator $V \mapsto -(\nabla_V^{\mathbf{g}} T_{\mathbf{g}})(\frac{D\mathbf{g}}{d\theta}\gamma')$ from the space of \mathcal{C}^2 -vector fields to the space of \mathcal{C}^0 -vector fields is compact, as it is continuous relatively to the \mathcal{C}^0 -topology, and the inclusion $\mathcal{C}^2 \hookrightarrow \mathcal{C}^0$ is compact. Hence, $\partial_1(\delta f)(\gamma, \mathbf{g})$ is Fredholm of index 0.

- If $\gamma \in \mathfrak{M}'$, then the orbit $\mathbb{S}^1 \cdot \gamma$ is a \mathcal{C}^1 -submanifold of \mathfrak{M} which is diffeomorphic to \mathbb{S}^1 . The tangent space $T_\gamma(\mathbb{S}^1 \cdot \gamma) \subset T_\gamma \mathfrak{M}$ is spanned by the tangent field γ' . Nondegeneracy of a critical orbit thus corresponds to the nondegeneracy of the closed geodesic.

The above observations guarantee that Theorem 3.2 applies to the closed geodesic variational problem setup, proving Proposition 4.4. \square

4.3. Harmonic maps. As a final geometric application of Theorem 3.2, let us consider the variational problem of harmonic maps between Riemannian manifolds. Let (M, \mathbf{g}) and $(\overline{M}, \overline{\mathbf{g}})$ be Riemannian manifolds. A \mathcal{C}^2 -map $\phi: M \rightarrow \overline{M}$ is said to be $(\mathbf{g}, \overline{\mathbf{g}})$ -harmonic if

$$(4.4) \quad \Delta_{\mathbf{g}, \overline{\mathbf{g}}}(\phi) := \text{tr}(\widehat{\nabla} d\phi) = 0$$

where $\widehat{\nabla}$ is the connection on the vector bundle $TM^* \otimes \phi^*(T\overline{M})$ over M induced by the Levi-Civita connections ∇ of \mathbf{g} and $\overline{\nabla}$ of $\overline{\mathbf{g}}$. The differential operator $\Delta_{\mathbf{g}, \overline{\mathbf{g}}}$ is the *Laplacian* relative to the metrics \mathbf{g} and $\overline{\mathbf{g}}$. Harmonic maps form a class that contains several geometrically important objects, see [12]. For instance, if $\dim M = 1$, harmonic maps $\phi: M \rightarrow \overline{M}$ are the geodesics of \overline{M} . In particular, setting $M = \mathbb{S}^1$, the previous statements in Subsection 4.2 regarding closed geodesics of \overline{M} can be reobtained (in the Riemannian case). The harmonic variational problem is also related to the CMC problem described in Subsection 4.1. Namely, an isometric immersion $\phi: M \rightarrow \overline{M}$ is minimal if and only if it is harmonic. In addition, setting $\overline{M} = \mathbb{R}$, harmonic maps are simply harmonic functions on M ; and setting $\overline{M} = \mathbb{S}^1$, harmonic maps are canonically identified with the harmonic 1-forms on M with integral periods. Henceforth, we assume compactness of the source manifold M to use the classic variational characterization of harmonic maps.

Let \mathfrak{M} be the Banach manifold $\mathcal{C}^{2,\alpha}(M, \overline{M})$ consisting of all maps $\phi: M \rightarrow \overline{M}$ that satisfy the $\mathcal{C}^{2,\alpha}$ -Hölder condition. Let Λ be the open subset of the Banach space of symmetric $(0, 2)$ -tensors of class \mathcal{C}^k on M , with $k \geq 3$, consisting of all positive-definite tensors, i.e., elements of Λ are Riemannian metric tensors of class \mathcal{C}^k on M . Set $\mathfrak{f}: \mathfrak{M} \times \Lambda \rightarrow \mathbb{R}$,

$$\mathfrak{f}(\phi, \mathbf{g}) = \frac{1}{2} \int_M \|d\phi(x)\|_{HS}^2 \text{vol}_{\mathbf{g}},$$

where $\text{vol}_{\mathbf{g}}$ is the volume form (or density) of \mathbf{g} and $\|d\phi(x)\|_{HS}$ is the Hilbert-Schmidt norm of the linear map $d\phi(x)$. This norm is defined by $\|d\phi(x)\|_{HS}^2 = \sum_i \|d\phi(x)e_i\|^2$, where $(e_i)_i$ an orthonormal basis of $T_x M$ relatively to \mathbf{g} and $\|\cdot\|$ is the norm on $T_{\phi(x)}\overline{M}$ induced by $\overline{\mathbf{g}}$. Equivalently, $\|d\phi(x)\|_{HS}^2 = \text{tr}(d\phi(x)^* \cdot d\phi(x))$.

For a given $\mathbf{g}_0 \in \Lambda$, critical points of the map $\phi \mapsto \mathfrak{f}(\phi, \mathbf{g}_0)$ are precisely the $(\mathbf{g}_0, \overline{\mathbf{g}})$ -harmonic maps $\phi: M \rightarrow \overline{M}$. For $\phi \in \mathfrak{M}$, the tangent space $T_\phi \mathfrak{M}$ is identified with the Banach space $\mathcal{C}^{2,\alpha}(\phi^* T\overline{M})$ of all $\mathcal{C}^{2,\alpha}$ -Hölder vector fields along ϕ . Given a such $V \in T_\phi \mathfrak{M}$, the derivative $\partial_1 \mathfrak{f}(\phi, \mathbf{g})V$ is given by:

$$(4.5) \quad \partial_1 \mathfrak{f}(\phi, \mathbf{g})V = \int_M \text{tr}(d\phi^* \overline{\nabla} V) \text{vol}_{\mathbf{g}}$$

$$= \int_M [\operatorname{div}(\mathrm{d}\phi^*(V)) - \bar{\mathbf{g}}(\Delta_{\mathbf{g}, \bar{\mathbf{g}}}(\phi), V)] \operatorname{vol}_{\mathbf{g}} \stackrel{\text{Stokes}}{=} - \int_M \bar{\mathbf{g}}(\Delta_{\mathbf{g}, \bar{\mathbf{g}}}(\phi), V) \operatorname{vol}_{\mathbf{g}},$$

where the trace is meant on the entries $\mathrm{d}\phi^*(\cdot)\bar{\nabla}_{(\cdot)}V$. The correspondent *Jacobi operator* J along a $(\mathbf{g}, \bar{\mathbf{g}})$ -harmonic map ϕ is the linear differential operator:

$$(4.6) \quad J_\phi(V) = -\Delta V + \operatorname{tr}(\bar{R}(\mathrm{d}\phi(\cdot), V)\mathrm{d}\phi(\cdot)),$$

defined in $\mathcal{C}^{2,\alpha}(\phi^*T\bar{M})$. Here \bar{R} is the curvature tensor of $\bar{\mathbf{g}}$, and ΔV is a vector field along ϕ uniquely defined by

$$(4.7) \quad \bar{\mathbf{g}}(\Delta V, W) = \operatorname{div}(\bar{\nabla}V^*)W - \bar{\mathbf{g}}(\bar{\nabla}V, \bar{\nabla}W), \quad W \in \mathcal{C}^{2,\alpha}(\phi^*T\bar{M})$$

i.e., $\Delta V(x) = \sum_i (\bar{\nabla}_{e_i}\bar{\nabla}V) e_i$, where $(e_i)_i$ is an orthonormal basis of T_xM . Analogously to the geodesic variational problem, a vector field V that satisfies $J_\phi(V) = 0$ will be called a *Jacobi field*. Observe that if K is a Killing vector field, then $J_\phi(K \circ \phi) = 0$. A $(\mathbf{g}, \bar{\mathbf{g}})$ -harmonic map $\phi: M \rightarrow \bar{M}$ is said to be *nondegenerate* if all Jacobi fields along ϕ are of this form.

Let G be the isometry group $\operatorname{Iso}(\bar{M}, \bar{\mathbf{g}})$ of the target manifold, acting on \mathfrak{M} by left-composition. Clearly, the functional \mathfrak{f} is invariant in the first variable under this action.¹⁰ Using results from [21], it is possible to prove that this action is smooth, since it is given by left-composition with smooth maps (see also [23] for the non-compact case). As a consequence, part of the technical arguments in Theorem 3.2 to deal with low regularity assumptions are not necessary in this context. Obviously, two harmonic maps $\phi_1, \phi_2: M \rightarrow \bar{M}$ are in the same $\operatorname{Iso}(\bar{M}, \bar{\mathbf{g}})$ -orbit if there exists an isometry $\psi: \bar{M} \rightarrow \bar{M}$ such that $\phi_2 = \psi \circ \phi_1$. In this case, ϕ_1 and ϕ_2 are called *geometrically equivalent*.

In this context, we may give the following statement that follows from the equivariant implicit function theorem.

Proposition 4.5. *Let $(\bar{M}, \bar{\mathbf{g}})$ be a smooth Riemannian manifold, let \mathbf{g} be a \mathcal{C}^k Riemannian metric tensor on the manifold M , with $k \geq 3$, and let $\phi: M \rightarrow \bar{M}$ be a $(\mathbf{g}, \bar{\mathbf{g}})$ -harmonic map. Consider Λ an open subset of a Banach space of symmetric $(0, 2)$ -tensors of class \mathcal{C}^k on M , with $\mathbf{g} \in \Lambda$, such that every tensor in Λ is a Riemannian metric tensor on M . Suppose ϕ is nondegenerate, i.e., all Jacobi fields along ϕ are of the form $\bar{K} \circ \phi$, where \bar{K} is a Killing vector field of \bar{M} . Then there exists a neighborhood \mathcal{U} of \mathbf{g} in Λ , a neighborhood \mathcal{V} of ϕ in $\mathcal{C}^{2,\alpha}(M, \bar{M})$, and a \mathcal{C}^{k-1} function $\mathcal{U} \ni \mathbf{h} \mapsto \phi_{\mathbf{h}} \in \mathcal{C}^{2,\alpha}(M, \bar{M})$ such that:*

- (a) $\phi_{\mathbf{h}}$ is an $(\mathbf{h}, \bar{\mathbf{g}})$ -harmonic map for all $\mathbf{h} \in \mathcal{U}$;
- (b) if $\mathbf{h} \in \mathcal{U}$ and $\varphi \in \mathcal{V}$ is an $(\mathbf{h}, \bar{\mathbf{g}})$ -harmonic map, then φ is geometrically equivalent to $\phi_{\mathbf{h}}$.

Proof. All assumptions of Theorem 3.2 are satisfied by the harmonic maps problem, using the following objects:

¹⁰One should observe that the harmonic map functional is also invariant under the action of the isometry group $\operatorname{Iso}(M, \mathbf{g})$ of the source manifold (M, \mathbf{g}) , which acts by right-composition in the space of maps from M to \bar{M} . However, equivariance with respect to such action will not be considered here. Namely, observe that, as the metric \mathbf{g} varies, then clearly also the group $\operatorname{Iso}(M, \mathbf{g})$ varies; thus, in order to deal with such equivariance, a formulation of the equivariant implicit function theorem for varying groups is needed. The assumption of equivariant nondegeneracy in Proposition 4.5 restricts the result to the case where (M, \mathbf{g}) has discrete isometry group or, more generally, when given any Killing field K of (M, \mathbf{g}) , then the field $\mathrm{d}\phi(K)$ along ϕ is the restriction to $\phi(M)$ of some Killing field \bar{K} of $(\bar{M}, \bar{\mathbf{g}})$.

- \mathfrak{M}' coincides with $\mathfrak{M} = \mathcal{C}^{2,\alpha}(M, \overline{M})$;
- \mathcal{E} is the mixed vector bundle whose fiber \mathcal{E}_ϕ is $\mathcal{C}^{0,\alpha}(\phi^*T\overline{M})$, the Banach space of all $\mathcal{C}^{0,\alpha}$ -Hölder vector fields along ϕ , endowed with the topology $\mathcal{C}^{2,\alpha}$ on the base and $\mathcal{C}^{0,\alpha}$ on the fibers;
- i is the inclusion, and j is induced by the L^2 -pairing that uses the inner product induced by $\overline{\mathbf{g}}$, and integrals taken with respect to the volume form (or density) of some fixed auxiliary¹¹ Riemannian metric \mathbf{g}_* on M ;
- given $\phi: M \rightarrow \overline{M}$ of class $\mathcal{C}^{2,\alpha}$ and a Riemannian metric tensor \mathbf{g} on M , $\delta f(\phi, \mathbf{g})$ is given by $-\zeta_{\mathbf{g}} \cdot \Delta_{\mathbf{g}, \overline{\mathbf{g}}}(\phi)$, where $\zeta_{\mathbf{g}}: M \rightarrow \mathbb{R}^+$ is the positive \mathcal{C}^k function satisfying $\zeta_{\mathbf{g}} \cdot \text{vol}_{\mathbf{g}_*} = \text{vol}_{\mathbf{g}}$, see (4.4) and (4.5);
- $\mathcal{Y} = T\mathfrak{M}$, κ is the identity map and \tilde{j} is induced by the L^2 -pairing, as above;
- identifying the Lie algebra \mathfrak{g} with the space of (complete) Killing vector fields on $(\overline{M}, \overline{\mathbf{g}})$, for $\phi \in \mathfrak{M}$, the map $d\beta_\phi(1): \mathfrak{g} \rightarrow T_\phi\mathfrak{M}$ associates to a Killing vector field \overline{K} the vector field $\overline{K} \circ \phi$ along ϕ ;
- given a $(\mathbf{g}, \overline{\mathbf{g}})$ -harmonic map $\phi: M \rightarrow \overline{M}$, the vertical projection of the derivative $\partial_1(\delta f)(\phi, \mathbf{g})$ is identified with $\zeta_{\mathbf{g}} \cdot J_\phi$, where J_ϕ is the Jacobi operator in (4.6). This is an elliptic second order partial differential operator and $\zeta_{\mathbf{g}} \cdot J_\phi: \mathcal{C}^{2,\alpha}(\phi^*T\overline{M}) \rightarrow \mathcal{C}^{0,\alpha}(\phi^*T\overline{M})$ is a Fredholm operator of index zero, see [25, Thm 1.1, (2)].

□

APPENDIX A. LOCAL ACTIONS

It is useful to have a version of the equivariant implicit function theorem for local actions of Lie groups on manifold. Once more, the paradigmatic example to keep in mind is the CMC embedding problem, in which one has a local action of the isometry group of the target manifold on a neighborhood of 0 of a Banach space, see Subsection 4.1. One observes that, given the local character of the result, the proof of Theorem 3.2 carries over to the case where the action of the Lie group G is only locally defined, in a sense that will be clarified in this appendix.

Let G be a Lie group and let \mathfrak{M} be a topological manifold. By a *local action* of G on \mathfrak{M} we mean a continuous map $\rho: \text{Dom}(\rho) \subset G \times \mathfrak{M} \rightarrow \mathfrak{M}$, defined on an open subset $\text{Dom}(\rho) \subset G \times \mathfrak{M}$ containing $\{1\} \times \mathfrak{M}$ satisfying:

- (a) $\rho(1, x) = x$ for all $x \in \mathfrak{M}$;
- (b) $\rho(g_1, \rho(g_2, x)) = \rho(g_1 g_2, x)$ whenever both sides of the equality are defined, i.e., for all those $x \in \mathfrak{M}$ and $g_1, g_2 \in G$ such that $(g_2, x) \in \text{Dom}(\rho)$, $(g_1, \rho(g_2, x)) \in \text{Dom}(\rho)$ and $(g_1 g_2, x) \in \text{Dom}(\rho)$.

The particular case of actions is when the domain $\text{Dom}(\rho)$ coincides with the entire $G \times \mathfrak{M}$. Local actions can be restricted, in the sense that, given any open subset \mathcal{A} of $\text{Dom}(\rho)$ containing $\{1\} \times \mathfrak{M}$, then the restriction $\rho|_{\mathcal{A}}$ of ρ to \mathcal{A} is again a local action of G on \mathfrak{M} .

Given a local action ρ of G on \mathfrak{M} , for $g \in G$, let ρ_g denote the map $\rho(g, \cdot)$, defined on an open (possibly empty) set $\text{Dom}(\rho_g) = \text{Dom}(\rho) \cap \{g\} \times \mathfrak{M}$. The following properties follow easily from the definition:

Lemma A.1. *Let $\rho: \text{Dom}(\rho) \subset G \times \mathfrak{M} \rightarrow \mathfrak{M}$ be a local action of G on M . Then:*

¹¹Note that one cannot use the volume form of \mathbf{g} in order to define j , because this metric is *variable* in the problem that we are considering.

- (i) for all $g \in G$, the map $\rho_g: \rho_g^{-1}(\text{Dom}(\rho_{g^{-1}})) \rightarrow \rho_{g^{-1}}^{-1}(\text{Dom}(\rho_g))$ is a homeomorphism;
- (ii) the set $\{(g, x) \in G \times \mathfrak{M} : x \in \rho_g^{-1}(\text{Dom}(\rho_{g^{-1}}))\}$ is an open subset that contains $\{1\} \times \mathfrak{M}$; in particular:
- (iii) for all $x \in \mathfrak{M}$, there exists an open neighborhood U_x of 1 in G such that for all $g \in U_x$, $x \in \rho_g^{-1}(\text{Dom}(\rho_{g^{-1}}))$.

In view of (iii), one can define a map $\beta_x: \text{Dom}(\beta_x) \subset G \rightarrow \mathfrak{M}$ on a neighborhood $\text{Dom}(\beta_x)$ of 1 in G , by $\beta_x(g) = \rho(g, x)$, cf. (3.1). In particular, if $x \in \mathfrak{M}$ is such that β_x is differentiable (at 1), then one has a well-defined linear map $d\beta_x(1): \mathfrak{g} \rightarrow T_x\mathfrak{M}$. A subset $C \subset \mathfrak{M}$ will be called G -invariant if $x \in C$ implies $\rho(g, x) \in C$ for all $g \in \text{Dom}(\beta_x)$.

Theorem A.2. *Theorem 3.2 holds when one replaces (A2) with the assumption that a local action ρ of G on \mathfrak{M} is given, and (A3) with the assumption that \mathfrak{f} satisfies:*

$$\mathfrak{f}(\rho(g, x), \lambda) = \mathfrak{f}(x, \lambda), \quad \text{for all } (g, x) \in \text{Dom}(\rho).$$

In this situation, the conclusion is that there exist open subsets $\Lambda_0 \subset \Lambda$ and $\mathfrak{M}_0 \subset \mathfrak{M}$, with $\lambda_0 \in \Lambda_0$ and $x_0 \in \mathfrak{M}_0$ and a C^k map $\sigma: \Lambda_0 \rightarrow \mathfrak{M}_0$ such that, for $(x, \lambda) \in \mathfrak{M}_0 \times \Lambda_0$, the identity $\partial_1 \mathfrak{f}(x, \lambda) = 0$ holds if and only if there exists $g \in G$, with $(\phi(\lambda), g) \in \text{Dom}(\rho)$ such that $x = \rho(g, \sigma(\lambda))$.

Proof. The proof of Theorem 3.2 carries over to this case, with minor modifications. One constructs a submanifold $S \subset \mathfrak{M}$ through x_0 with the property that $T_{x_0}S \oplus \text{Im}(d\beta_{x_0}(1)) = T_{x_0}\mathfrak{M}$, and considers the restriction of \mathfrak{f} to the product $S \times \Lambda$. The only question that needs an extra argument here is to show that the set $\rho((G \times S) \cap \text{Dom}(\rho))$ is a neighborhood of x_0 ; this is the analogue of claim (2) in the proof of Theorem 3.2 (see page 12). Once again, this follows as an application of Proposition 3.4, used in the following setup: A is an open neighborhood of 1 in G , M is an open neighborhood of x_0 in \mathfrak{M} , these open subsets being chosen in such a way that the product $A \times M$ be contained in the open set $\{(g, x) \in G \times \mathfrak{M} : x \in \rho_g^{-1}(\text{Dom}(\rho_{g^{-1}}))\}$, see part (ii) of Lemma A.1. Set $N = \mathfrak{M}$, $P = S$, $a_0 = 1$ and $m_0 = x_0$; the function χ is the restriction of ρ to $A \times M$. The conclusion of Proposition 3.4 says that for all $x \in M$, there exists $g \in A$ such that $x \in \rho_g^{-1}(\text{Dom}(\rho_{g^{-1}}))$ and $\rho(g, x) \in S$; since $\rho(g, x) \in \text{Dom}(\rho_{g^{-1}})$, then $x = \rho(g^{-1}, \rho(g, x)) \in \rho((G \times S) \cap \text{Dom}(\rho))$, i.e., $\rho((G \times S) \cap \text{Dom}(\rho))$ contains the open subset $M \ni x_0$. The rest of the proof of Theorem 3.2 can now be repeated *verbatim*. \square

APPENDIX B. INVARIANT VOLUME FUNCTIONALS

A technical assumption made in Proposition 4.1 concerns the existence of a generalized volume functional \mathcal{V} which is invariant under left-composition with isometries of $(\overline{M}, \overline{\mathbf{g}})$. Let us consider a compact differentiable manifold M , possibly with boundary ∂M , and a Riemannian manifold $(\overline{M}, \overline{\mathbf{g}})$, with $m = \dim(M) = \dim(\overline{M}) - 1$.

Definition B.1. Let \mathcal{U} be an open subset of embeddings $x: M \hookrightarrow \overline{M}$. An *invariant volume functional* on \mathcal{U} is a real valued function $\mathcal{V}: \mathcal{U} \rightarrow \mathbb{R}$ satisfying:

- (a) $\mathcal{V}(x) = \int_M x^*(\eta)$, where η is an m -form defined on an open subset $U \subset \overline{M}$ such that $d\eta$ is equal to the volume form $\text{vol}_{\overline{\mathbf{g}}}$ of $\overline{\mathbf{g}}$ in U ;
- (b) given $x \in \mathcal{U}$, for all ϕ isometry of $(\overline{M}, \overline{\mathbf{g}})$ sufficiently close to the identity it is $\mathcal{V}(\phi \circ x) = \mathcal{V}(x)$.

If M has boundary, the invariance property (b) is required to hold only for isometries ϕ near the identity, and preserving $x(\partial M)$, i.e., $\phi(x(\partial M)) = x(\partial M)$.

By (b), the generalized volume $\mathcal{V}(x)$ does not depend on the parameterization of x , i.e., $\mathcal{V}(x \circ \psi) = \mathcal{V}(x)$ for all diffeomorphisms ψ of M . Hence, \mathcal{V} defines a smooth map (in local charts) in a neighborhood of x in the set of unparameterized embeddings of M into \overline{M} , see [1].

We now give some basic examples of invariant volume functionals.

Example 3. Assume that the image $x(M)$ of x is the boundary of a bounded open subset Ω of \overline{M} ; note that this is an open condition in the set of unparameterized embeddings.¹² For y sufficiently close to x in the \mathcal{C}^1 -topology, one has $y(M) = \partial\Omega_y$ for some open bounded subset Ω_y of \overline{M} . Setting $\mathcal{V}(y) = \text{vol}(\Omega_y)$, i.e., the volume of this open bounded subset, we have an invariant volume functional \mathcal{V} (use Stokes' theorem). In fact, when $x(M) = \partial\Omega$, any function \mathcal{V} satisfying (a) of Definition B.1 coincides with such volume functional, by Stokes' theorem.

Example 4. Assume that \overline{M} is non-compact, so that its volume form $\text{vol}_{\overline{\mathbf{g}}}$ is exact, and assume that $G = \text{Iso}(\overline{M}, \overline{\mathbf{g}})$ is compact. Let η be a primitive of $\text{vol}_{\overline{\mathbf{g}}}$, and set:

$$\eta^G = \int_G \phi^*(\eta) d\phi,$$

the integral being taken relatively to a Haar measure of volume 1 of G . Then, η^G is a primitive of $\text{vol}_{\overline{\mathbf{g}}}$ which is G -invariant. In particular, the functional $\mathcal{V}(x) = \int_M x^*(\eta^G)$ is an invariant volume functional.

Using the compactness argument above, we obtain immediately:

Corollary B.2. Assume that M is a compact manifold with (non-empty) boundary ∂M , and that \overline{M} is non-compact. Given an embedding $x: M \hookrightarrow \overline{M}$, there exists a volume functional \mathcal{V} defined in the set:

$$\left\{ y: M \hookrightarrow \overline{M} \text{ embedding} : y(\partial M) = x(\partial M) \right\}$$

which is invariant under all isometries ϕ that preserve $x(\partial M)$.

Proof. The natural action of the isometry group $G = \text{Iso}(\overline{M}, \overline{\mathbf{g}})$ on \overline{M} is proper; since ∂M is compact, the closed subgroup G_0 of G consisting of isometries ϕ satisfying $\phi(x(\partial M)) = x(\partial M)$ is compact. If η is a primate of $\text{vol}_{\overline{\mathbf{g}}}$ in \overline{M} , then one can average η on G_0 , obtaining a G_0 -invariant primitive η^{G_0} of $\text{vol}_{\overline{\mathbf{g}}}$. Clearly, the volume functional defined by η^{G_0} is also G_0 -invariant. \square

Non-compactness of the ambient manifold \overline{M} and compactness of its isometry group is a rather restrictive assumption; we will now determine more general conditions that guarantee the existence of invariant volume functionals.

¹²If x is transversely oriented, then such condition is equivalent to the fact that x induces the null map in homology $H^1(M) \rightarrow H^1(\overline{M}, M \setminus x(M))$. In particular, the condition is stable by \mathcal{C}^1 -small perturbations of x .

Lemma B.3. *Let $U \subset \overline{M}$ be an open subset and let η be any primitive of $\text{vol}_{\overline{g}}$ on U . Consider the volume functional $\mathcal{V}(y) = \int_M y^*(\eta)$, defined on the set of embeddings $\mathfrak{M}(U)$ of M into \overline{M} having image in U , and let*

$$\rho: \text{Dom}(\rho) \subset \text{Iso}(\overline{M}, \overline{g}) \times \mathfrak{M}(U) \longrightarrow \mathfrak{M}(U)$$

be the natural local action by left-composition with isometries of $(\overline{M}, \overline{g})$.

Then, for all $\phi \in \text{Iso}(\overline{M}, \overline{g})$, the map $y \mapsto \mathcal{V}(y) - \mathcal{V}(\phi \circ y)$ is locally constant on $\text{Dom}(\rho_\phi)$. If $\eta - \phi^(\eta)$ is exact in U and $x \in \mathfrak{M}(U)$, then for ϕ sufficiently close to the identity, $\mathcal{V}(\phi \circ x) = \mathcal{V}(x)$.*

Proof. For all $\phi \in \text{Iso}(\overline{M}, \overline{g})$, $\phi^*(\eta)$ is a primitive of the volume form $\text{vol}_{\overline{g}}$, so that $\eta - \phi^*(\eta)$ is closed in its domain. If $x, y \in \mathfrak{M}(U)$ are C^0 -close, then x and y are homotopic, hence, using Stokes' theorem, if x and y are in $\text{Dom}(\rho_\phi)$, $\int_M x^*(\eta - \phi^*(\eta)) = \int_M y^*(\eta - \phi^*(\eta))$, i.e., $\mathcal{V}(x) - \mathcal{V}(\phi \circ x) = \mathcal{V}(y) - \mathcal{V}(\phi \circ y)$.

If $\eta - \phi^*(\eta)$ is exact, then so is $x^*(\eta - \phi^*(\eta))$, thus $\int_M x^*(\eta - \phi^*(\eta)) = 0$, i.e., $\mathcal{V}(x) = \mathcal{V}(\phi \circ x)$. Note that this equality holds also when M has boundary, provided that $\phi(x(\partial M)) = x(\partial M)$. \square

Lemma B.3 suggests how to construct invariant volume functionals. The natural setup consists of a pair of open subset $U_1 \subset U_2 \subset \overline{M}$, an m -form η on U_2 which is a primitive of $\text{vol}_{\overline{g}}$ in U_2 , with the following properties:

- $\phi(U_1) \subset U_2$ for ϕ in a neighborhood of the identity in $\text{Iso}(\overline{M}, \overline{g})$;
- $\eta - \phi^*(\eta)$ is exact in U_1 for $\phi \in \text{Iso}(\overline{M}, \overline{g})$ near the identity.

Corollary B.4. *Given objects U_1, U_2 and η as above, the map $\mathcal{V}(x) = \int_M x^*(\eta)$ is an invariant volume functional in the set of embeddings $x: M \hookrightarrow \overline{M}$ having image contained in U_1 .*

Proof. It follows immediately from the last statement of Lemma B.3. \square

Notice that $\eta - \phi^*(\eta)$ is closed, hence if U_1 has vanishing de Rham cohomology in dimension m , then it is exact. This observation provides a large class of examples of manifolds $(\overline{M}, \overline{g})$ where it is possible to define local invariant volume functionals.

Example 5. *If \overline{M} is a non-compact manifold whose m^{th} de Rham cohomology space is zero, then the volume functional defined by any primitive of $\text{vol}_{\overline{g}}$ is invariant under the (whole) isometry group. More generally, if $x: M \hookrightarrow \overline{M}$ is an embedding that has image contained in an open subset whose n^{th} de Rham cohomology space is zero, then there exists a volume functional invariant under isometries near the identity, defined in an open neighborhood \mathcal{U} of x in the set of embeddings of M into \overline{M} . In particular, this applies when \overline{M} is \mathbb{R}^{m+1} or $\overline{M} = S^{m+1}$. Manifolds of the form $\overline{M}^{m+1} = \mathbb{R}^k \times N^{m+1-k}$, $k \geq 1$, have trivial m^{th} de Rham cohomology space. Manifolds of the form $\overline{M}^{m+1} = S^k \times N^{m+1-k}$, $k \geq 1$, have open dense subsets with trivial m^{th} de Rham cohomology space.*

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